# Hypothesis Testing

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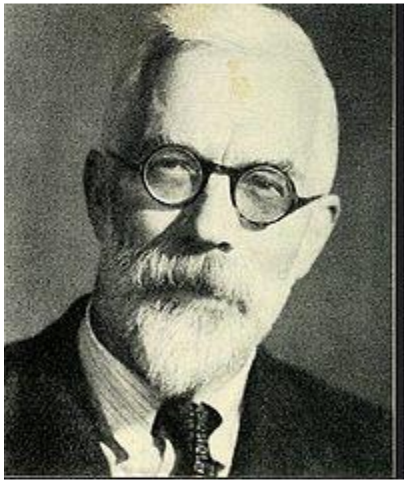

**Ronald Fisher**
**1935**

# Hypothesis Testing

Illustrative Example

# Introduction

- Basic idea
  - » Perform statistical analysis of data to determine if a particular hypothesis should be accepted
  - » Use results of the hypothesis test for decision making
- General procedure
  - » Formulate the hypothesis
  - » Formulate an alternative to the hypothesis
  - » Choose a significance level $\alpha$ that represents the probability of rejecting a true hypothesis
  - » Perform statistical analysis on samples to test the validity of the hypothesis
  - » Either accept or reject the hypothesis based on the test

# Illustrative Example

- A company intends to purchase a lot of 100 solar cells if the manufacturer's claim that the film thickness is 200 microns can be verified
- The null hypothesis (hypothesis) is that the film thickness $\mu = \mu_0 = 200$ microns
- The alternative hypothesis (alternative) is that $\mu = \mu_1 < \mu_0$
- The hypothesis will be accepted if it is satisfied with a probability $\alpha$ (significance level)
- Otherwise the hypothesis will be rejected and the solar cells will not be purchased
- The decision needs to be made with a small number of samples

# Illustrative Example

- The thicknesses of $n$ = 25 cells are measured to yield $\bar{x}$ = 197 microns and $s$ = 6 microns
- The question is if the sample mean $\bar{x}$ is significantly different than the desired value $\mu_0$
- If the thicknesses are assumed to be normally distributed, then $T$ follows a $t$-distribution with $n = m$-1 degrees of freedom:

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

- The sample mean $\bar{x}$ and standard deviation $s$ are observed values of $\bar{X}$ and $S$
- Select a significance level α = 5%

# Illustrative Example

- Need to find $c$ such that: $P(T \le c) = \alpha = 0.05$
- Because the t-distribution is symmetric, the $c$ value can be obtained from Table A9:

$$P(T \le \tilde{c}) = 1 - \alpha = 0.95 \qquad \Rightarrow \quad \tilde{c} = 1.71 \quad \Rightarrow \quad c = -\tilde{c} = -1.71$$

$$m = n - 1 = 24$$

- If the hypothesis is true, then there is only a 5% chance that an observed value $t$ of $T$ will have a value [-∞,-1.71]

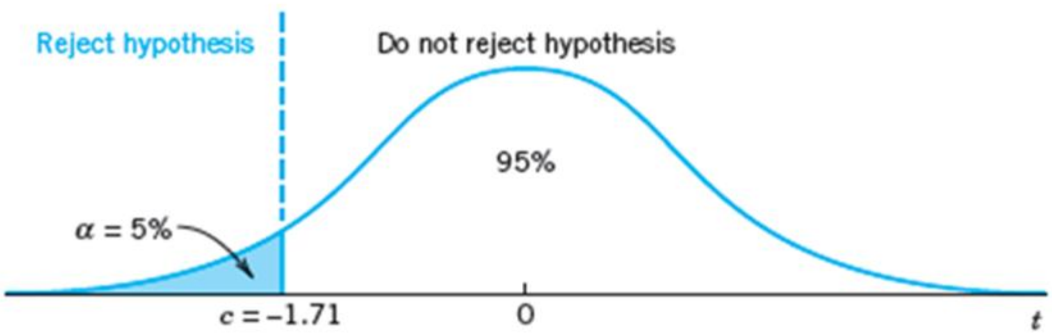


Fig. 532. t-distribution in Example 1

# Illustrative Example

- Compute the *t* statistic from the samples:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{197 - 200}{6/\sqrt{25}} = -2.5$$

- Because t = -2.5 < c = -1.71:
  - » The null hypothesis that $\mu = \mu_0$ = 200 microns is rejected
  - » The alternative hypothesis that $\mu = \mu_1$ < 200 microns is accepted
  - » The solar cells are not purchased

# Illustrative Example

- The likelihood of accepting the hypothesis will increase if:
  - » The sample mean $\bar{x}$ is closer to the hypothesized mean $\mu_0$
  - » The sample standard deviation $s$ increases
  - » The number of samples $n$ decreases
  - » The significance level $\alpha$ decreases

| $\bar{x}$ | s | n | α | m | c | t | Hypothesis |
|---|---|---|---|---|---|---|---|
| 197 | 6 | 25 | 0.05 | 24 | -1.71 | -2.50 | Reject |
| **198** | 6 | 25 | 0.05 | 24 | -1.71 | -1.67 | Accept |
| 197 | **10** | 25 | 0.05 | 24 | -1.71 | -1.5 | Accept |
| 197 | 6 | **10** | 0.05 | 9 | -1.83 | -1.58 | Accept |
| 197 | 6 | 25 | **0.005** | 24 | -2.80 | -2.50 | Accept |

# Hypothesis Testing

Alternatives and Testing Errors

# Distribution for Mean Hypothesis Testing

- Let $\{X_1,\ldots,X_n\}$ be independent normal random variables, each with the same mean μ and variance $\sigma^2$
- Then the random variable $T$ follows a t-distribution with $m = n$-1:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \quad \bar{X} = \frac{1}{n}(X_1 + \cdots + X_n) \quad S^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2$$

- The t-distribution is tabulated in Table A9 where values of $z$ are given as a function of values of:
  - » The cumulative distribution function $F(z)$
  - » The degrees of freedom $m$

# Distribution for Variance Hypothesis Testing

- Let $\{X_1,\ldots,X_n\}$ be independent normal random variables, each with the same mean $\mu$ and variance $\sigma^2$

- Then the random variable $Y$ follows a chi-squared distribution with $m = n$-1:

$$Y = (n-1)\frac{S^2}{\sigma^2} \qquad \bar{X} = \frac{1}{n}(X_1 + \cdots + X_n) \qquad S^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \bar{X})^2$$

- The chi-squared distribution is tabulated in Table A10 where values of $z$ are given as a function of values of:
  - » The cumulative distribution function $F(z)$
  - » The degrees of freedom $m$

# One-Sided and Two-Sided Alternatives

- Let $\theta$ be an unknown parameter in a distribution for which it is hypothesized that $\theta = \theta_0$
- Alternatives:

$\theta > \theta_0$

$\theta < \theta_0$

$\theta \neq \theta_0$

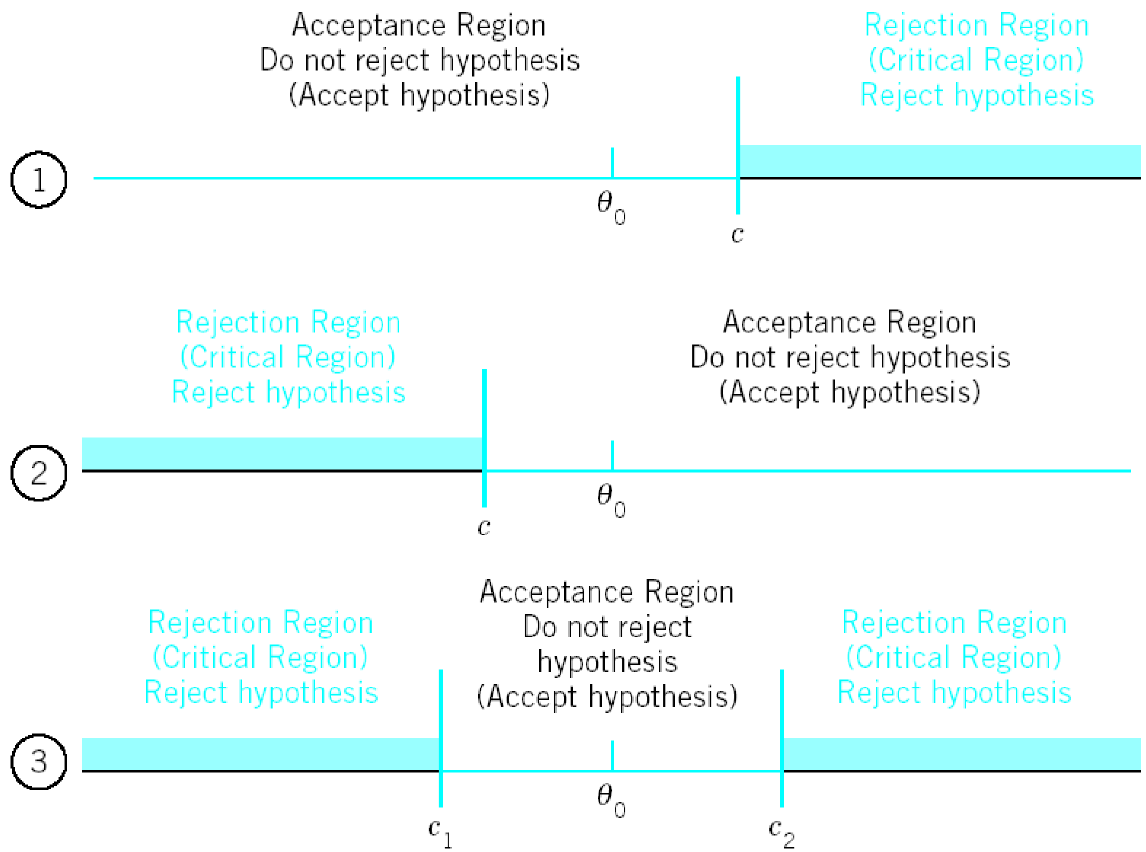


**Fig. 532.** Test in the case of alternative (1) (upper part of the figure), alternative (2) (middle part), and alternative (3)

# Hypothesis Testing Errors

- Hypothesis testing involves a risk of making false decisions
- Type I error – reject a true hypothesis
  - » $\alpha$ = probability of making a Type I error
- Type II error – accept a false hypothesis
  - » $\beta$ = probability of making a Type II error

**Table 25.4** **Type I and Type II Errors in Testing a Hypothesis $\theta = \theta_0$ Against an Alternative $\theta = \theta_1$**

| | | Unknown Truth | |
|---|---|---|---|
| | | $\theta = \theta_0$ | $\theta = \theta_1$ |
| Accepted | $\theta = \theta_0$ | True decision<br>$P = 1 - \alpha$ | Type II error<br>$P = \beta$ |
| | $\theta = \theta_1$ | Type 1 error<br>$P = \alpha$ | True decision<br>$P = 1 - \beta$ |

# Hypothesis Testing Errors

- Type I and II testing errors are conflicting requirements
- If the significance level $\alpha$ decreases, then $\beta$ increases and the chance of accepting a false hypothesis increases
- The text shows how to calculate $\beta$ from $\alpha$ and the critical value *c*

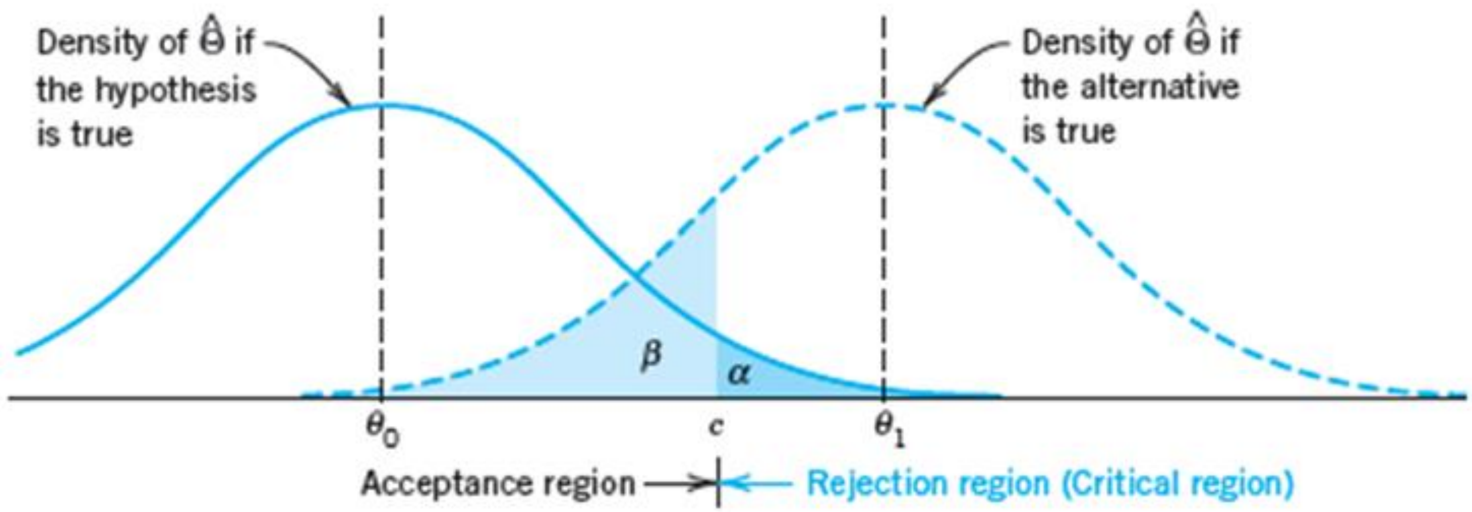


**Fig. 534.** Illustration of Type I and II errors in testing a hypothesis $\theta = \theta_0$ against an alternative $\theta = \theta_1$ ($> \theta_0$, right-sided test)

# Hypothesis Testing

Mean and Variance Hypothesis Tests

# Mean Test for Normal Distributions

- Data: $n$ random samples $\{x_1, x_2, \ldots, x_n\}$
- Method shown below is for left-handed test
- Hypothesis: mean is $\mu_0$ instead of $\mu_1 < \mu_0$
- Select significance level $\alpha$
- Compute observed value of $T$ as:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \quad \bar{x} = \frac{1}{n}(x_1 + \cdots + x_n) \quad s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2$$

- Determine $c$ from Table A9 with $m = n$-1 as: $P(T < c) = \alpha$
- Accept hypothesis if $t > c$; otherwise reject hypothesis

# Mean Hypothesis Test Example

- Measurements of polymer molecular weight (scaled by $10^{-5}$)

  $\bar{x} = 1.258 \quad s^2 = 0.0049$

- Hypothesis: $\mu_0 = 1.3$ instead of the alternative $\mu_1 < \mu_0$
- Significance level: $\alpha = 0.10$
- Degrees of freedom: $m = 9$
- Critical value

$$P(T \leq \tilde{c}) = 1 - \alpha = 0.90 \quad \Rightarrow \quad \tilde{c} = 1.38$$

$$P(T \leq c) = \alpha = 0.10 \quad \Rightarrow \quad c = -\tilde{c} = -1.38$$

- Sample $t$

$$t = \frac{1.258 - 1.3}{\sqrt{0.0049} / \sqrt{10}} = -1.897 < c = -1.38$$

- Reject hypothesis

# Variance Test for Normal Distributions

- Data: $n$ random samples $\{x_1, x_2, \ldots, x_n\}$
- Method shown below is for right-handed test
- Hypothesis: variance is $\sigma_0^2$ instead of $\sigma_1^2 > \sigma_0^2$
- Compute sample variance as: $s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2$
- Select significance level $\alpha$
- Determine $c$ from Table A10 with $m = n$-1 as:

$$P(Y > c) = \alpha \quad \Rightarrow \quad P(Y \le c) = 1 - \alpha$$

- Compute critical value of $s^2$ as: $c^* = \sigma_0^2 c/(n-1)$
- Accept hypothesis if $s^2 < c^*$; otherwise reject hypothesis

# Variance Hypothesis Test Example

- Measurements of polymer molecular weight (scaled by $10^{-5}$)

  $$\bar{x} = 1.258 \quad s^2 = 0.0049$$

- Hypothesis: $\sigma_0^2 = 0.005$ instead of the alternative $\sigma_1^2 > \sigma_0^2$
  Significance level: $\alpha = 0.05$
- Degrees of freedom: $m = 9$
- Critical value

$$P(Y > c) = \alpha = 0.05 \quad \Rightarrow \quad P(Y \leq c) = 1 - \alpha = 0.95 \quad \Rightarrow \quad c = 16.92$$

$$c^* = \frac{\sigma_0^2 c}{n-1} = \frac{(0.005)(16.92)}{9} = 0.0094$$

- Since $s^2 < c^*$, the hypothesis is accepted

# Hypothesis Testing

In-class Exercise