Probability Theory

1. Basic concepts
2. Probability
3. Permutations and combinations
4. In-class exercise

Probability Theory

Basic Concepts

Statistical Thinking

- Experimental measurements can be viewed as variables that are subject to random variations
- This randomness cannot be predicted in a deterministic sense
- Such variables are known as random variables
- The chance of a random variable having a particular value is governed by probability theory
- Data can be thought of as random samples from an underlying, unknown probability distribution
- A goal of statistics is to extract information about the probability distribution from a usually small number of samples

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Mean and Variance

- Data: multiple measurements of the same quantity

$$\{x_1 \quad x_2 \quad \cdots \quad x_{n-1} \quad x_n\}$$

- Definitions
  - » Range: $R = x_{\max} - x_{\min}$
  - » Median: middle value when values are ordered according to their magnitudes
  - » Important properties:

Mean $\quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Variance $\quad s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2 \quad$ Standard deviation $\quad s = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2}$

  - » The mean represents the average value
  - » The variance is a measure of variability

Mean and Variance Example

- Data set 1

$$x = \{0.25 \quad 0.50 \quad 0.75 \quad 1 \quad 1.25 \quad 1.50 \quad 1.75\}$$

$$\bar{x} = \frac{1}{7}\sum_{j=1}^{7} x_j = 1$$

$$s^2 = \frac{1}{7-1}\sum_{j=1}^{7} (x_j - \bar{x})^2 = 0.29$$

- Data set 2

$$x = \{-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4\}$$

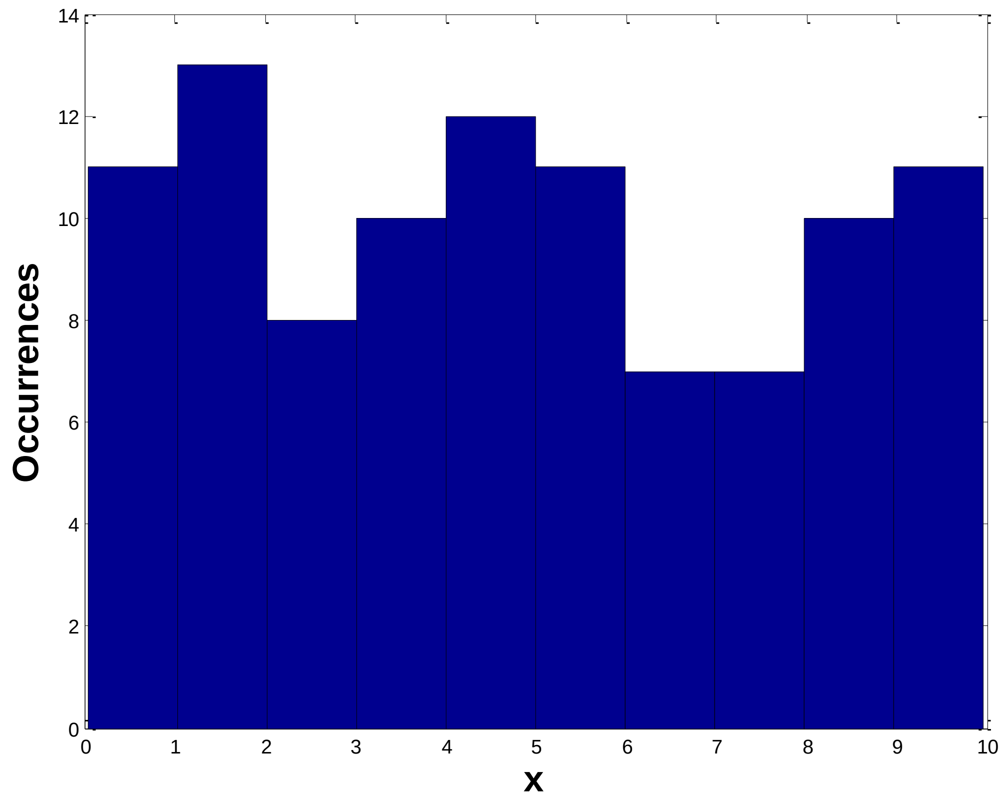
$$\bar{x} = \frac{1}{7}\sum_{j=1}^{7} x_j = 1$$

$$s^2 = \frac{1}{7-1}\sum_{j=1}^{7} (x_j - \bar{x})^2 = 4.67$$

- These two data sets are very different despite having the same mean

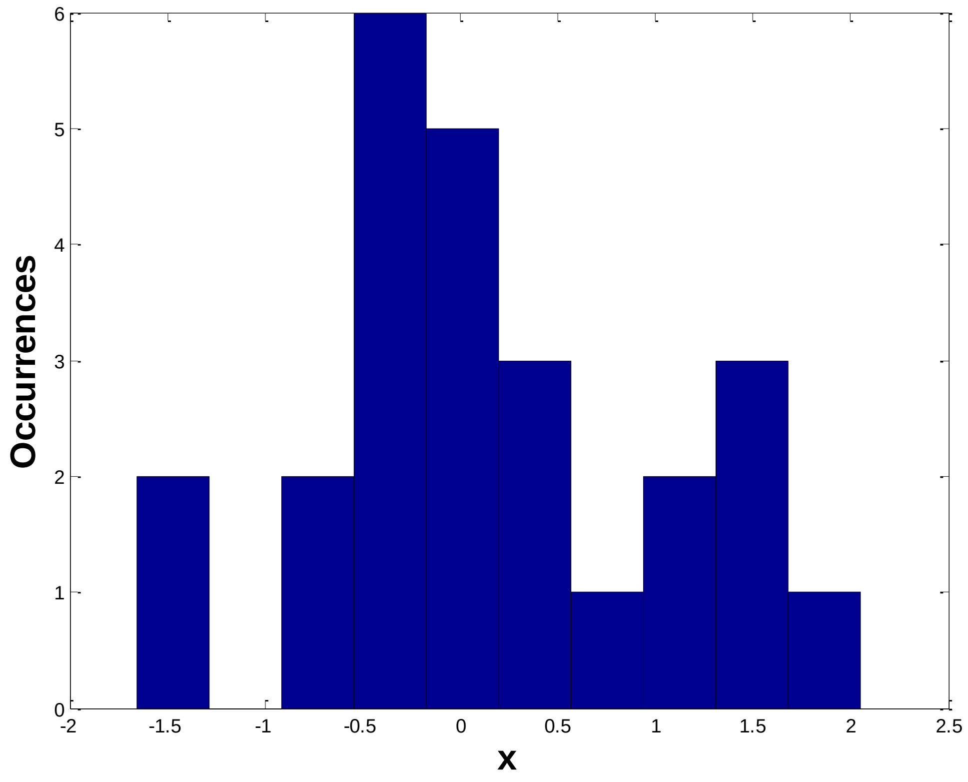
Histograms

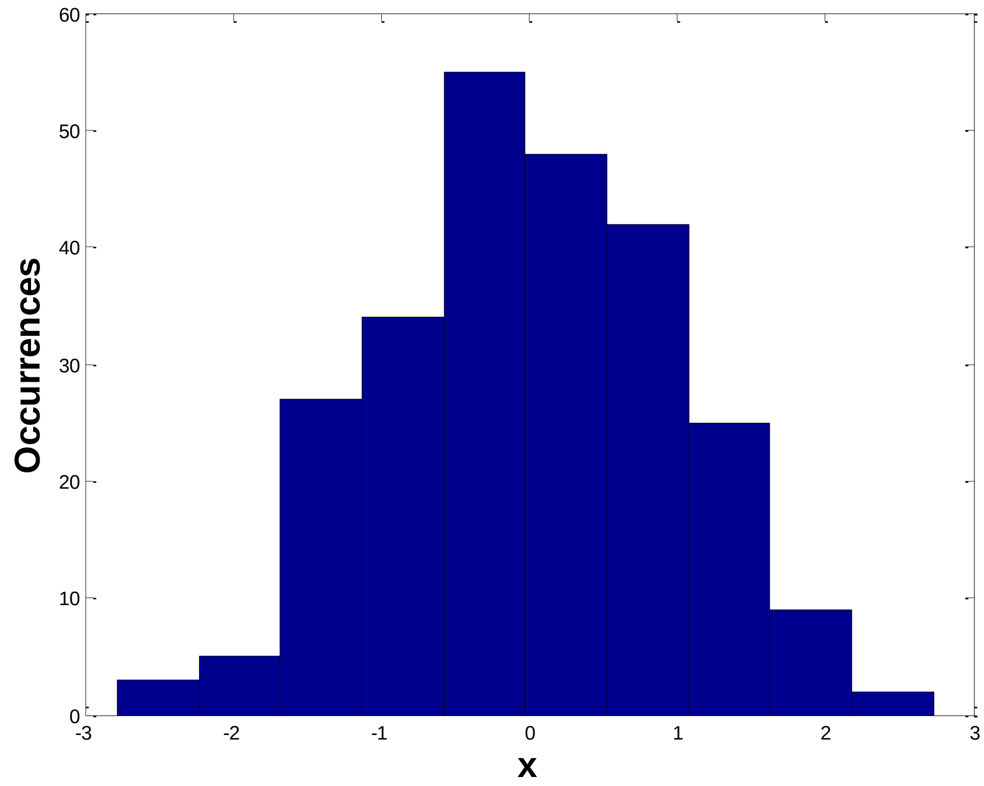
- Large data sets are conveniently represented graphically in histograms
- Data are binned into intervals based on their values
- The number of occurrences is plotted verses the bin values to represent the variability



Effect of Data Set Size

- Estimation of statistical properties is improved as the number of samples increases





25 samples
Mean = 0.16
Variance = 0.82

250 samples
Mean = -0.003
Variance = 0.92

Probability Theory

Probability

Probability Theory

- Experiment: process of measurement or observation
- Trial: performance of a single experiment
- Outcome: the result of a trial (also called a sample)
- Sample size n: the number of trials performed
- Sample space S: set of all possible outcomes
- Events: subsets of the sample space
- Key idea: each sample represents a value of the random variable

Probability Theory Example

- Experiment: measurement of polymer thin film thickness
- Trial: performance of a thickness measurement on a single thin film
- Outcome: the thickness measurement
- Sample size n: the number of thin films measured
- Sample space S = {too thin, acceptable, too thick}
- Events: too thin, acceptable, too thick
- But cannot realistically measure the thickness of every thin film manufactured

Events

- Sample space divided into events (A_1, A_2, A_3, …)
- Union and intersection of events

$$\text{Union} \quad A_1 \cup A_2 = \text{all points in either set}$$

$$\text{Intersection} \quad A_1 \cap A_2 = \text{all points in both sets}$$

$$S = A_1 \cup A_2 \cup A_3 \cup \cdots$$

- Disjoint and complement events

$$\text{Disjoint events} \quad A_1 \cap A_2 = \varnothing$$

$$\text{Complement events} \quad A_1 \cup A_1^C = S \quad A_1 \cap A_1^C = \varnothing$$

- Thin film example
  - » Three events: too thin, acceptable, too thick
  - » The union of the three events is the sample space
  - » The events have no intersection and are therefore disjoint
  - » Each event is the complement of the other two events

Definition of Probability

- Simple definition for finitely many equally likely outcomes:

$$P(A_j) = \frac{\text{number of points in } A_j}{\text{number of points in } S} \quad P(S) = 1$$

- Relative frequency for disjoint events:

$$f_{rel}(A_j) = \frac{\text{number of times } A_j \text{ occurs}}{n} \quad 0 \le f_{rel}(A_j) \le 1$$

$$f_{rel}(S) = 1 \quad f_{rel}(A_1 \cup A_2) = f_{rel}(A_1) + f_{rel}(A_2) \quad f_{rel}(A_1 \cap A_2) = 0$$

- $P(A_j)$ satisfies the following axioms of probability:

$$0 \le P(A_j) \le 1 \quad P(S) = 1$$

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) \cup P(A_2) \cup \cdots \quad P(A_1 \cap A_2 \cap \cdots) = P(A_1) \cap P(A_2) \cap \cdots = \varnothing$$

Basic Theorems of Probability

- Complementation: $P(A_j^C) = 1 - P(A_j)$
- Addition rule for mutually exclusive events

$$P(A_1 \cup A_2 \cup \cdots \cup A_m) = P(A_1) + P(A_2) + \cdots + P(A_m)$$

- Addition rule for arbitrary events

$$\boldsymbol{P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)}$$

- Conditional probability of A_2 given A_1

$$P(A_2 \mid A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)}$$

$$P(A_1 \cap A_2) = P(A_1)P(A_2 \mid A_1) = P(A_2)P(A_1 \mid A_2)$$

- Independent events

$$P(A_1 \cap A_2) = P(A_1)P(A_2) \quad P(A_2 \mid A_1) = P(A_2) \quad P(A_1 \mid A_2) = P(A_1)$$

Probability Examples

- Probability that at least one coin will turn heads up from five tossed coins
  - » Number of outcomes: $2^5 = 32$
  - » Probability of each outcome: 1/32
  - » Probability of no heads: $P(A^C) = 1/32$
  - » Probability at least one head: $P(A) = 1\text{-}P(A^C) = 31/32$
- Probability of getting an odd number or a number less than 4 from a single dice toss
  - » Probability of odd number: $P(A) = 3/6$
  - » Probability of number less than 4: $P(B) = 3/6$
  - » Probability of both: $P(A \cap B) = 2/6$
  - » Probability of either:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 3/6 + 3/6 - 2/6 = 2/3$$

Probability Theory

Permutations and Combinations

Permutations

- Permutation – arrangement of objects in a particular order
- The number of permutations of n different objects taken all at a time is: $n! = 1 \cdot 2 \cdot 3 \cdots n$
- The number of permutations of n objects divided into c different classes taken all at a time is:

$$\frac{n!}{n_1! n_2! \cdots n_c!} \qquad n_1 + n_2 + \cdots + n_c = n$$

- Number of permutations of n different objects taken k at a time is:

Without repetitions $\quad n(n-1)(n-2)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$

With repetitions $\quad n^k$

Permutation Examples

- Box containing 6 red and 4 blue balls
  - » Compute probability that all red balls and then all blue balls will be removed
  - » $n_1 = 6$, $n_2 = 4$
  - » Probability

$$P = \frac{1}{n!/n_1!n_2!} = \frac{6!4!}{10!} \approx 0.005 = 0.5\%$$

- Coded telegram
  - » Letters arranged in five-letter words: $n = 26$, $k = 5$
  - » Total number of different words: $n^k = 26^5 = 11{,}881{,}376$
  - » Total number of different words containing each letter no more than once:

$$\frac{n!}{(n-k)!} = \frac{26!}{(26-5)!} = 7{,}893{,}600$$

Combinations

- Combination – selection of objects without regard to order
- Binomial coefficients

Real number a $$\binom{a}{k} \equiv \frac{a(a-1)(a-2)\cdots(a-k+1)}{k!}$$

Integer $0 \le k \le n$ $$\binom{n}{k} \equiv \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

- Number of combinations of n different objects taken k at a time is:

Without repetitions $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

With repetitions $$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}$$

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Combination Examples

- Effect of repetitions
  - » Three letters *a*, *b*, *c* taken two at a time ($n = 3$, $k = 2$)
  - » Combinations without repetition

$$\binom{n}{k} = \binom{3}{2} = \frac{3!}{2!(3-2)!} = 3 \quad ab \quad ac \quad bc$$

  - » Combinations with repetitions

$$\binom{n+k-1}{k} = \binom{4}{2} = \frac{4!}{2!(4-2)!} = 6 \quad \begin{matrix} ab & ac & bc \\ aa & bb & cc \end{matrix}$$

- 500 light bulbs taken 5 at a time
  - » Repetitions not possible
  - » Combinations $\displaystyle \binom{n}{k} = \binom{500}{5} = \frac{500!}{5!(500-5)!} = 255{,}244{,}686{,}600$

Probability Theory

In-class Exercise