Mixed Boundary Value ODEs

1. Mixed boundary value problems
2. Countercurrent heat exchanger
3. In-class exercise
4. Shooting solution method

Theophil Hildebrandt
1918

Wallie Hurwitz
1921

Issai Schur
1921

Mixed Boundary Value ODEs

Mixed boundary value problems

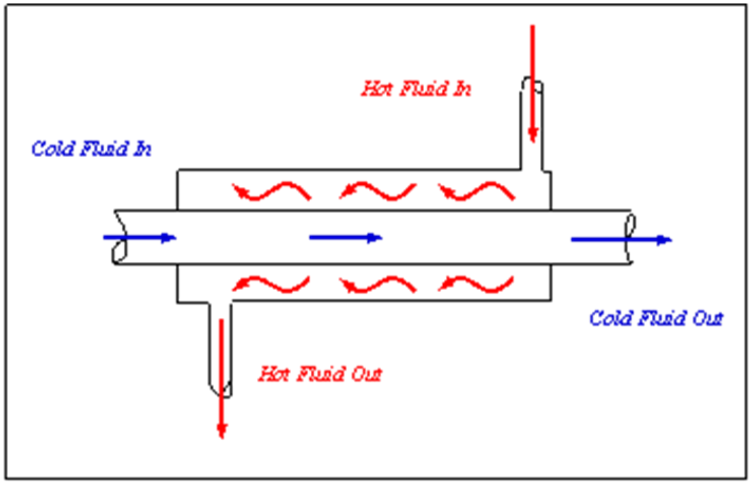
Mixed Boundary Value ODEs

- All the ODE systems we have considered thus far are initial value problems (IVPs)

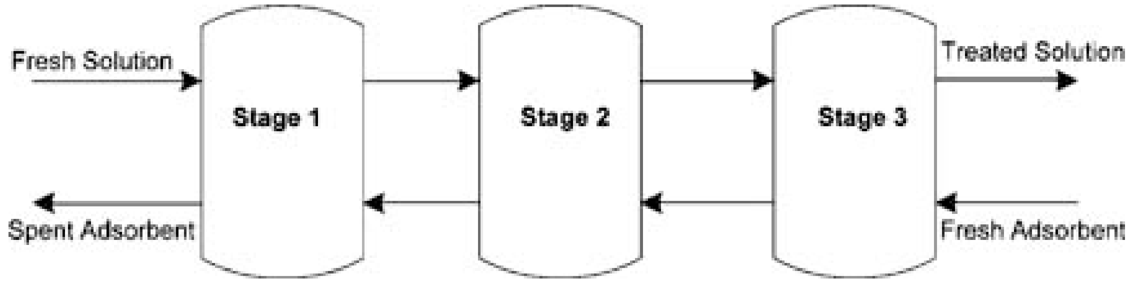
$$\frac{d\mathbf{y}}{dz} = \mathbf{f}(x, \mathbf{y}) \quad \mathbf{y}(0) = \mathbf{y}_0 \qquad\qquad \frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}) \quad \mathbf{y}(0) = \mathbf{y}_0$$

- IVPs are solved by integrating the ODEs forward in space or time using the initial condition specified at $z = 0$ or $t = 0$
- ODE systems in which boundary conditions are specified at two or more boundaries are known as mixed boundary values problems
- Mixed boundary values problems cannot be solved like IVPs

Chemical Engineering Examples

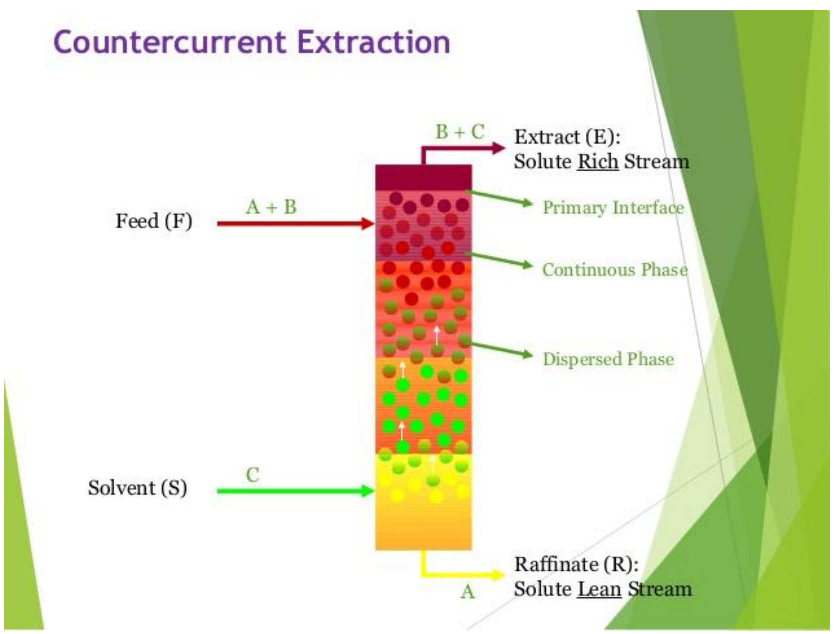


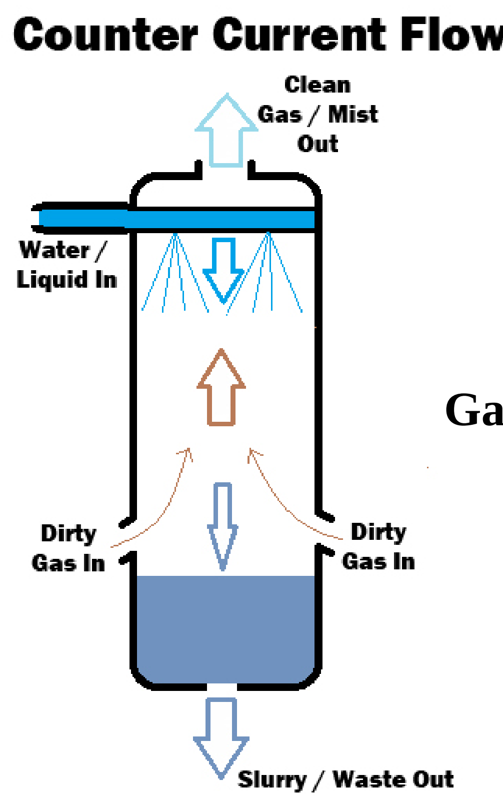
Heat exchanger



Adsorber

Extracter





Gas scrubber

Mixed Boundary Value ODEs

- A simple mixed boundary value problem (MVBP)

$$\frac{dy}{dz} = f_1(z, y, w) \quad y(0) = y_0 \qquad \frac{dw}{dz} = f_2(z, y, w) \quad w(L) = w_L$$

- Boundary conditions are specified at the two ends of the domain [0, *L*]
- The first equation needs to be integrated from 0 to *L*
- The second equation needs to be integrated rom *L* to 0
- Different numerical solution methods are needed for this class of ODEs

A Linear Example

- Mixed boundary value problem (MVBP)

$$\frac{dy}{dz} = -y + 2w \quad y(0) = 1 \qquad \frac{dw}{dz} = 2y - w \quad w(1) = 2$$

- Matrix representation

$$\frac{d}{dz}\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} y \\ w \end{bmatrix} \qquad \begin{bmatrix} y(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- Eigenvalues and eigenvectors

$$\lambda_1 = 1 \quad \mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_2 = -3 \quad \mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A Linear Example

- Solution form

$$\mathbf{x}(z)=\begin{bmatrix} y(z) \\ w(z) \end{bmatrix}=c_1\mathbf{x}^{(1)}e^{\lambda_1 z}+c_2\mathbf{x}^{(2)}e^{\lambda_2 z}=c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{z}+c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-3z}$$

- Apply boundary conditions

$$\begin{bmatrix} y(0) \\ w(1) \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}=c_1\begin{bmatrix} e^{0} \\ e^{1} \end{bmatrix}+c_2\begin{bmatrix} e^{0} \\ -e^{-3} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 0.74 \\ 0.26 \end{bmatrix}$$

- Final solution

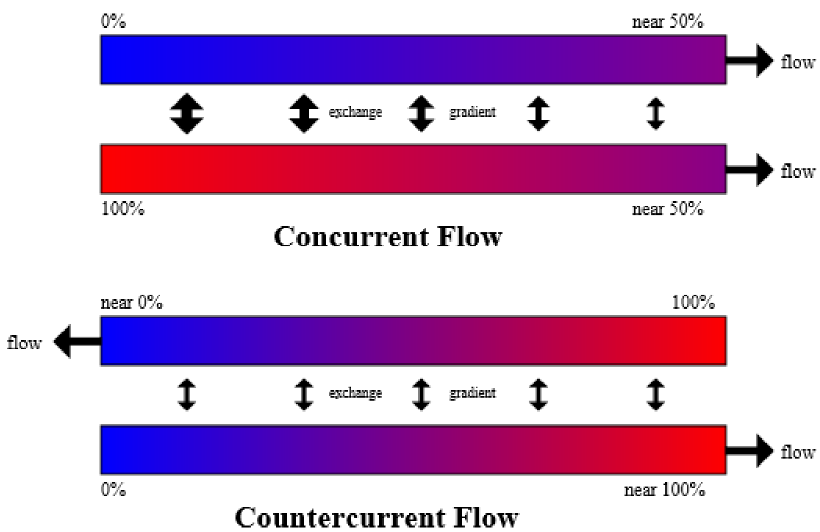
$$\mathbf{x}(z)=\begin{bmatrix} y(z) \\ w(z) \end{bmatrix}=0.74\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{z}+0.26\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-3z}$$

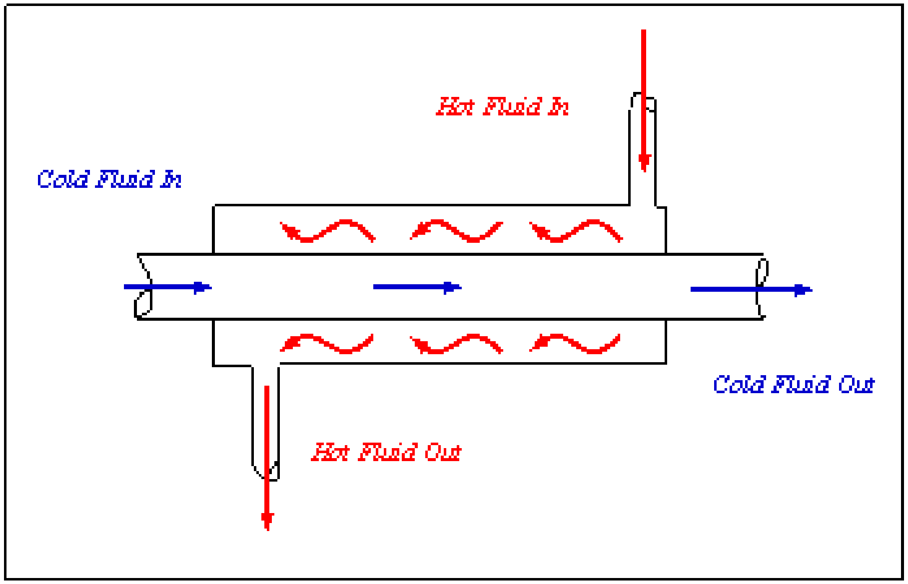
Mixed Boundary Value ODEs

Countercurrent Heat Exchanger

Countercurrent Heat Exchanger

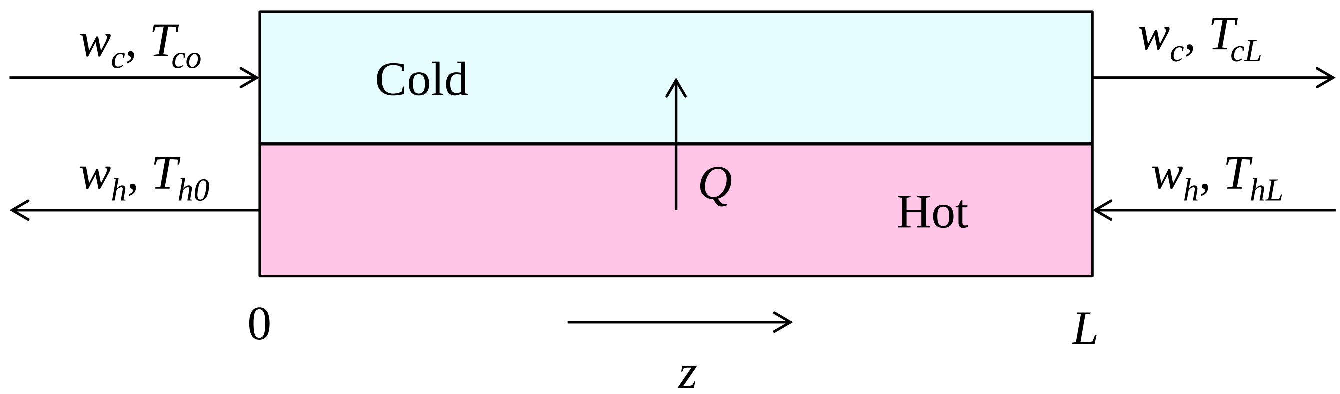
- Heat exchangers are used to transfer heat from one flowing fluid to another flowing fluid.
- The rate of heat transfer depends on the local temperature difference between the two fluids.
- A larger average temperature difference can be achieved with countercurrent flow of the two fluids
- The temperatures of the fluids at their respective entrance points represent mixed boundary values



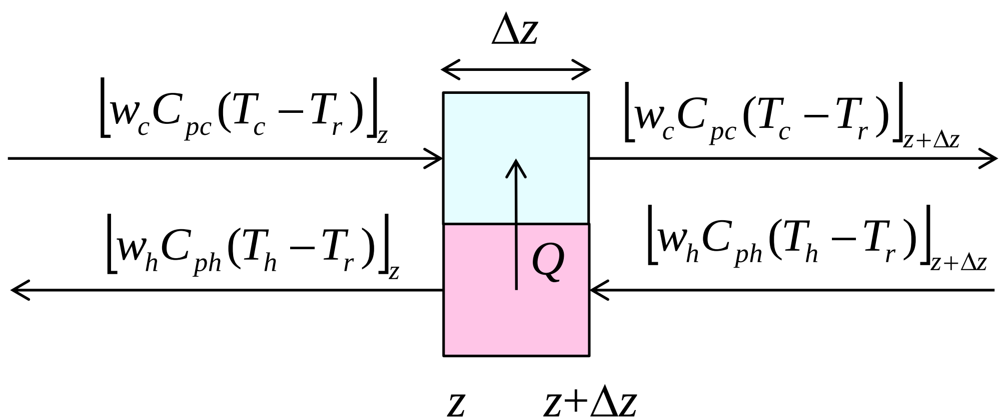


Countercurrent Heat Exchanger

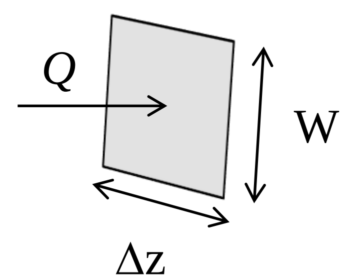
- Process schematic



- Differential element for enthalpy balance



- Heat flux from hot stream to cold stream



$$Q = UA(T_h - T_c) = U\Delta z W(T_h - T_c)$$

$$A = W\Delta z$$

Countercurrent Heat Exchanger

- Energy balance on cold stream

$$\underbrace{0}_{\text{Energy accumulation}} = \underbrace{\left[w_c C_{pc} (T_c - T_r) \right]_z}_{\text{Energy in}} - \underbrace{\left[w_c C_{pc} (T_c - T_r) \right]_{z+\Delta z}}_{\text{Energy out}} + \underbrace{UW\Delta z (T_h - T_c)}_{\text{Energy transfer in}}$$

$$0 = \frac{-\left\{ \left[w_c C_{pc} (T_c - T_r) \right]_{z+\Delta z} - \left[w_c C_{pc} (T_c - T_r) \right]_z \right\}}{\Delta z} + UW(T_h - T_c)$$

- Take the limit as Δz → 0

$$0 = -\frac{d\left[w_c C_{pc} (T_c - T_r) \right]}{dz} + UW(T_h - T_c)$$

$$0 = -w_c C_{pc} \frac{dT_c}{dz} + UW(T_h - T_c) \qquad T_c(0) = T_{c0}$$

Countercurrent Heat Exchanger

- Energy balance on hot stream

$$\underbrace{0}_{\text{Energy accumulation}} = \underbrace{\left[w_h C_{ph}(T_h - T_r)\right]_{z+\Delta z}}_{\text{Energy in}} - \underbrace{\left[w_h C_{hc}(T_h - T_r)\right]_z}_{\text{Energy out}} - \underbrace{UW\Delta z(T_h - T_c)}_{\text{Energy transfer out}}$$

$$0 = -w_h C_{ph}\frac{dT_h}{dz} + UW(T_h - T_c) \quad T_h(L) = T_{hL}$$

- Mixed value boundary problem

$$\frac{dT_c}{dz} = \frac{UW}{w_c C_{pc}}(T_h - T_c) \quad T_c(0) = T_{c0}$$

$$\frac{dT_h}{dz} = \frac{UW}{w_h C_{ph}}(T_h - T_c) \quad T_h(L) = T_{hL}$$

Mixed Boundary Value ODEs

In-class Exercise

Mixed Boundary Value ODEs

Shooting Solution Method

The Shooting Method

- A simple mixed boundary value problem (MBVP)

$$\frac{dy}{dz} = f_1(z, y, w) \quad y(0) = y_0 \qquad\qquad \frac{dw}{dz} = f_2(z, y, w) \quad w(L) = w_L$$

- Analytical solution is not possible if the ODEs are nonlinear
- The objective of the shooting method is to convert the MBVP to an IVP
- The unknown boundary value $w(0)$ is guessed
- Then the two coupled ODEs can be integrated simultaneously from 0 to L
- If $w(L) \neq w_L$, then a new guess is required
- The procedure is repeated until $w(L) \approx w_L$ to yield $y(z)$ and $w(z)$ that satisfy the specified boundary conditions
- There are rational ways to pick $w(0)$, but they are not discussed here

Heat Exchanger Example

- Countercurrent heat exchanger

$$\frac{dT_c}{dz} = \frac{UW}{w_c C_{pc}}(T_h - T_c) \quad T_c(0) = T_{c0}$$

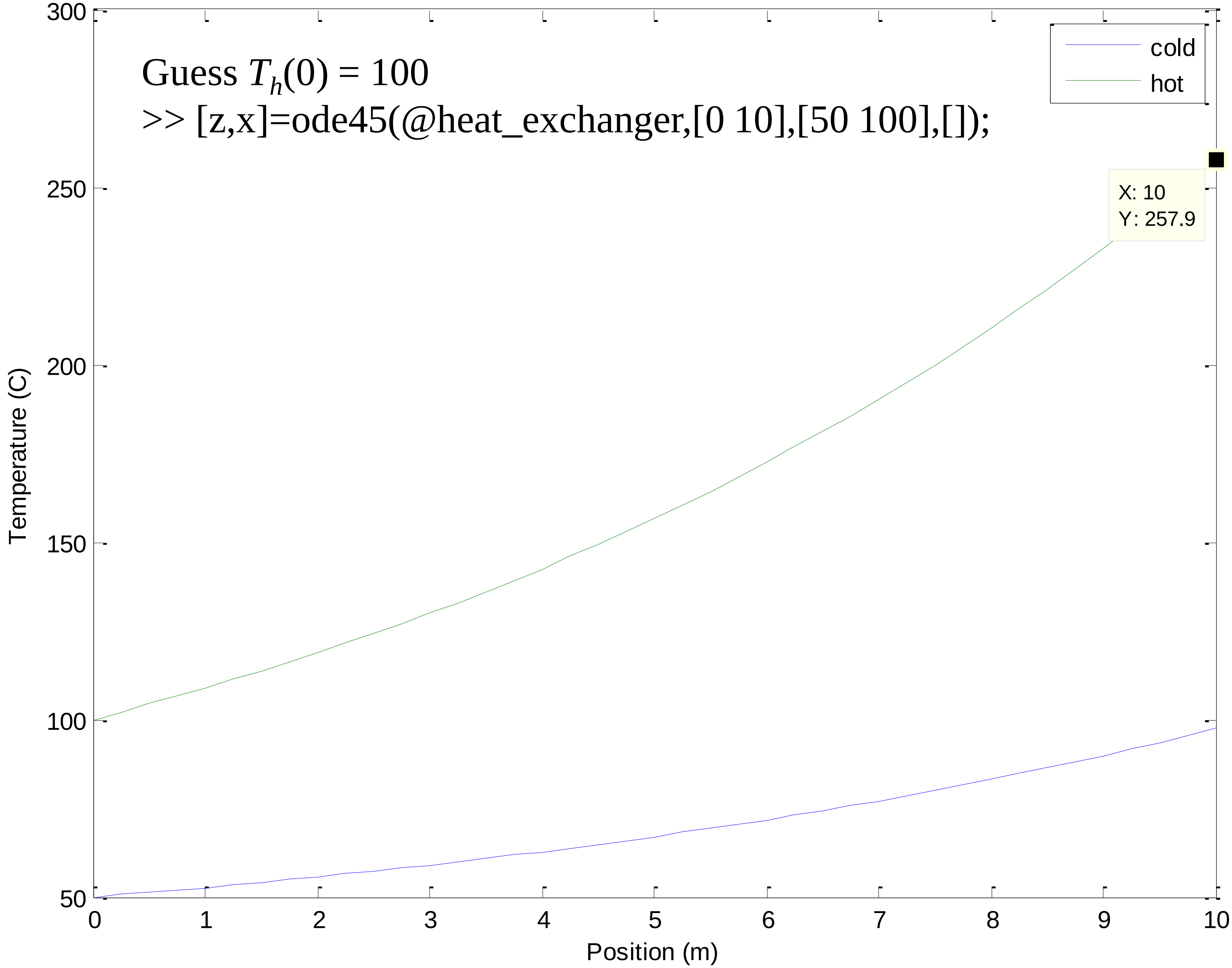
$$\frac{dT_h}{dz} = \frac{UW}{w_h C_{ph}}(T_h - T_c) \quad T_h(L) = T_{hL}$$

- Parameter values (all SI units): $L = 10$; $U = 100$, $W = 10$; $w_c = 5$, $C_{pc} = 4000$, $w_h = 1$, $C_{ph} = 6000$
- Boundary conditions: $T_{\mathrm{co}} = 50$, $T_{\mathrm{hL}} = 300$

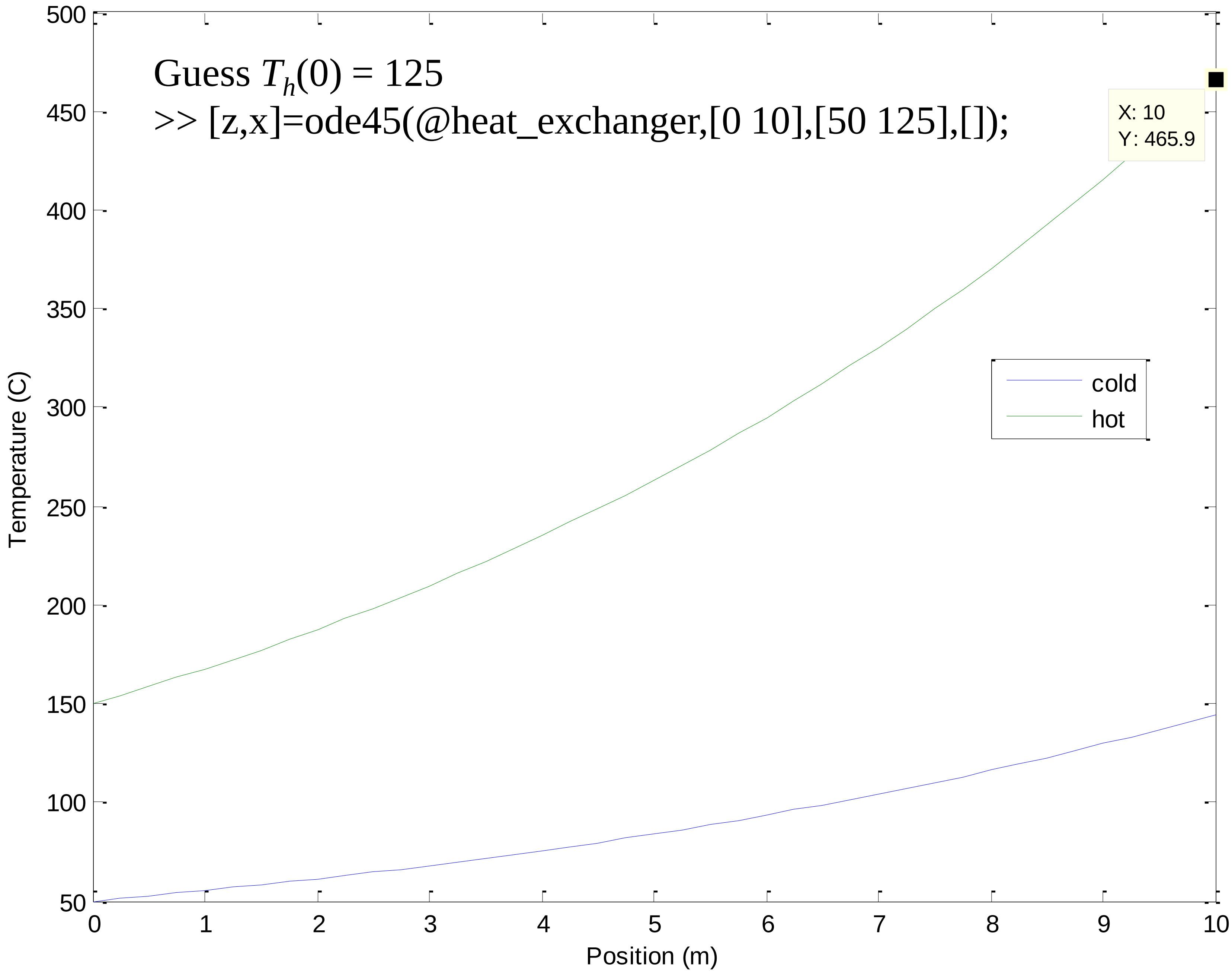
Heat Exchanger Example

```
function f = heat_exchanger(z,x)
wc = 5;
cpc = 4000;
wh = 1;
cph = 6000;
u = 100;
w = 10;
tc = x(1);
th = x(2);
f(1) = u*w/(wc*cpc)*(th-tc);
f(2) = u*w/(wh*cph)*(th-tc);
f = f';
```

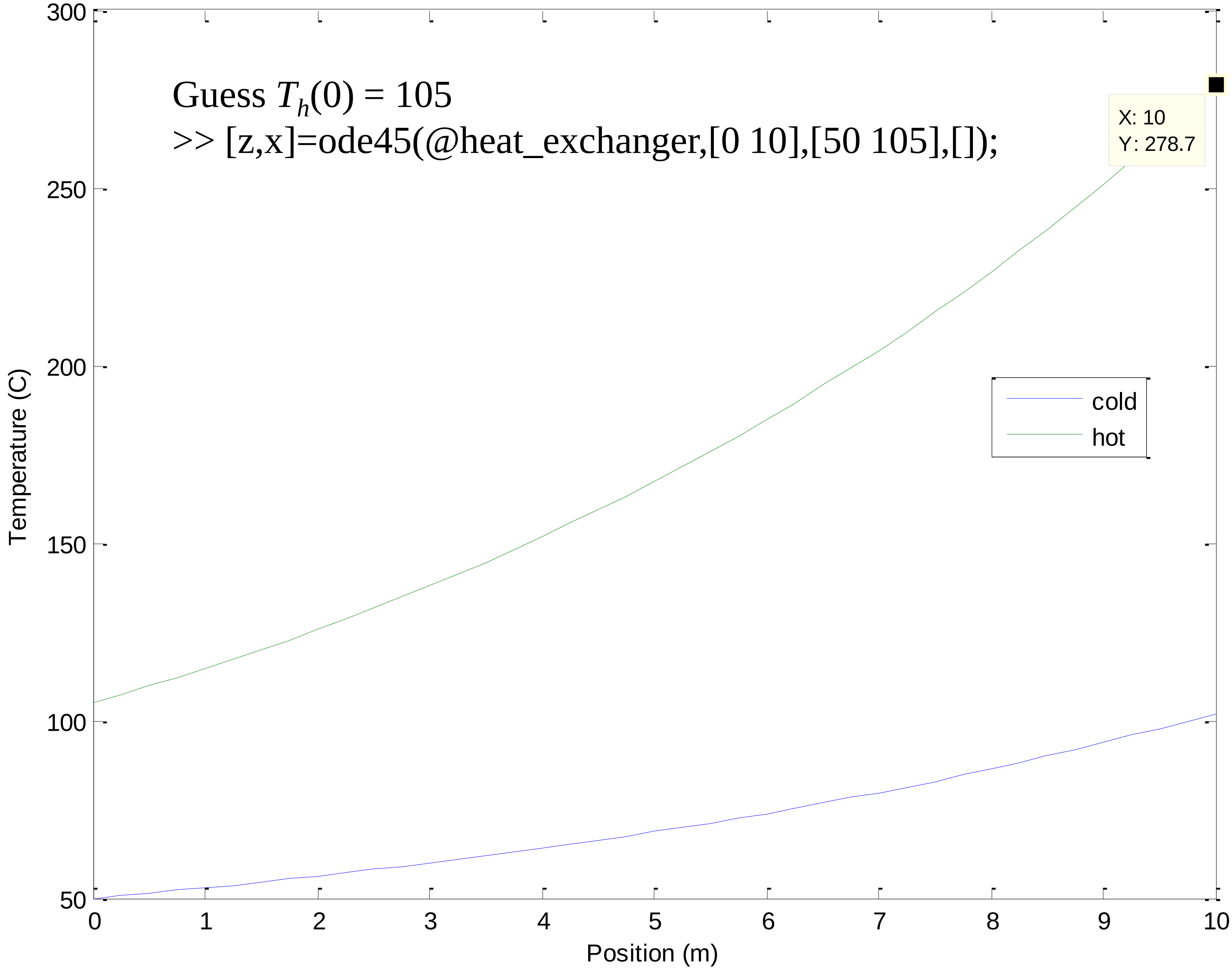
Heat Exchanger Example



Heat Exchanger Example



Heat Exchanger Example



Heat Exchanger Example

Guess $T_h(0) = 110.1$

```
>> [z,x]=ode45(@heat_exchanger,[0 10],[50 110.1],[]);
```