The Eigenvalue Problem

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The Eigenvalue Problem

Eigenvalues and Eigenvectors

The Eigenvalue Problem

- Consider a *n*x*n* matrix **A**
- Vector equation: $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$
  - » Seek solutions for **x** and λ
  - » λ satisfying the equation are called the eigenvalues
  - » Eigenvalues can be real and/or imaginary; distinct and/or repeated
  - » **x** satisfying the equation are called the eigenvectors
- Nomenclature
  - » The set of all eigenvalues is called the spectrum
  - » Absolute value of an eigenvalue:

$$\lambda_j = a + ib \quad \Rightarrow \quad |\lambda_j| = \sqrt{a^2 + b^2}$$

  - » The largest of the absolute values of the eigenvalues is called the spectral radius

Determining Eigenvalues

- Vector equation
  - » $\mathbf{Ax} = \lambda\mathbf{x} \rightarrow (\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = 0$
  - » $\mathbf{A}-\lambda\mathbf{I}$ is called the characteristic matrix
- Non-trivial solutions exist if and only if:

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

  - » This is called the characteristic equation
- Characteristic polynomial
  - » nth-order polynomial in λ
  - » Roots are the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

Eigenvalue Properties

- Eigenvalues of **A** and $\mathbf{A}^T$ are equal
- Singular matrix has at least one zero eigenvalue
- Eigenvalues of $\mathbf{A}^{-1}$: $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$
- Eigenvalues of diagonal and triangular matrices are equal to the diagonal elements
- Trace

$$Tr(\mathbf{A}) = \sum_{j=1}^{n} \lambda_j$$

- Determinant

$$|\mathbf{A}| = \prod_{j=1}^{n} \lambda_j$$

Determining Eigenvectors

- First determine eigenvalues: $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$
- Then determine the eigenvector corresponding to each eigenvalue:

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0 \quad \Rightarrow \quad (\mathbf{A}-\lambda_k\mathbf{I})\mathbf{x}_k=0$$

- Eigenvectors are unique up to a scalar multiple
- Distinct eigenvalues
  - » Produce linearly independent eigenvectors
- Repeated eigenvalues
  - » Produce linearly dependent eigenvectors
  - » Procedure to determine eigenvectors can be more cumbersome
- Complex eigenvalues
  - » Produce complex eigenvectors
  - » Procedure to determine eigenvectors requires complex algebra

The Eigenvalue Problem

Calculation Examples

Distinct Real Eigenvalue Example

- Characteristic matrix

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}1 & 2\\ 3 & -4\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1-\lambda & 2\\ 3 & -4-\lambda\end{bmatrix}$$

- Characteristic equation

$$\left|\mathbf{A}-\lambda\mathbf{I}\right|=(1-\lambda)(-4-\lambda)-(2)(3)=\lambda^2+3\lambda-10=0$$

- Eigenvalues: $\lambda_1 = -5$, $\lambda_2 = 2$

Distinct Real Eigenvector Example

- Eigenvalues

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \quad \begin{matrix} \lambda_1 = -5 \\ \lambda_2 = 2 \end{matrix}$$

- Determine eigenvectors: $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

$$\begin{matrix} x_1 + 2x_2 = \lambda x_1 \\ 3x_1 - 4x_2 = \lambda x_2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} (1-\lambda)x_1 + 2x_2 = 0 \\ 3x_1 - (4+\lambda)x_2 = 0 \end{matrix}$$

- Eigenvector for $\lambda_1 = -5$

$$\begin{matrix} 6x_1 + 2x_2 = 0 \\ 3x_1 + x_2 = 0 \end{matrix} \quad \Rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

- Eigenvector for $\lambda_1 = 2$

$$\begin{matrix} -x_1 + 2x_2 = 0 \\ 3x_1 - 6x_2 = 0 \end{matrix} \quad \Rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Repeated Real Eigenvalue Example

- Characteristic matrix

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}2 & 5\\ 0 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2-\lambda & 5\\ 0 & 2-\lambda\end{bmatrix}$$

- Characteristic equation

$$\left|\mathbf{A}-\lambda\mathbf{I}\right|=(2-\lambda)(2-\lambda)-(5)(0)=0$$

- Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 2$

Repeated Real Eigenvector Example

- Eigenvalues

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} \quad \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 2 \end{matrix}$$

- Determine eigenvectors: $\mathbf{Ax} = \lambda\mathbf{x}$

$$\begin{matrix} 2x_1 + 5x_2 = \lambda x_1 \\ 0x_1 + 2x_2 = \lambda x_2 \end{matrix} \quad \Rightarrow \quad \begin{matrix} (2-\lambda)x_1 + 5x_2 = 0 \\ 0x_1 + (2-\lambda)x_2 = 0 \end{matrix}$$

- Eigenvectors for $\lambda = 2$

$$\begin{matrix} 0x_1 + 5x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{matrix} \quad \Rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- Eigenvectors are linearly dependent

Complex Eigenvalue Example

- Characteristic matrix

$$\mathbf{A}-\lambda\mathbf{I}=\begin{bmatrix}-1 & -1\\ 1 & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-1-\lambda & -1\\ 1 & -1-\lambda\end{bmatrix}$$

- Characteristic equation

$$\left|\mathbf{A}-\lambda\mathbf{I}\right|=(-1-\lambda)(-1-\lambda)-(-1)(1)=(1+\lambda)^2+1=0$$

- Eigenvalues: λ_1 = -1+i, λ_2 = -1-i

Complex Eigenvector Example

- Eigenvalues

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i \end{aligned}$$

- Determine eigenvectors: **Ax** = λ**x**

$$\begin{aligned} -x_1 - x_2 &= \lambda x_1 \\ x_1 - x_2 &= \lambda x_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (-1-\lambda)x_1 - x_2 &= 0 \\ x_1 + (-1-\lambda)x_2 &= 0 \end{aligned}$$

- Eigenvector for λ = -1+i

$$\begin{aligned} -ix_1 - x_2 &= 0 \\ x_1 - ix_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

- Eigenvector for λ = -1-i

$$\begin{aligned} ix_1 - x_2 &= 0 \\ x_1 + ix_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The Eigenvalue Problem

In-class Exercise

The Eigenvalue Problem

Matrix Diagonalization

Similarity Transformations

- Eigenbasis
  - » If a nxn matrix has n distinct eigenvalues, the eigenvectors form a basis for R^n
  - » If a nxn matrix has repeated eigenvalues, the eigenvectors may not form a basis for R^n (see text)
- Similar matrices
  - » Two nxn matrices are similar if there exists a nonsingular nxn matrix **P** such that: $\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{AP}$
  - » Similar matrices have the same eigenvalues
  - » If **x** is an eigenvector of **A**, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of the similar matrix

Matrix Diagonalization

- Assume the *n*x*n* matrix **A** has an eigenbasis
- Form the *n*x*n* modal matrix **X** with the eigenvectors of **A** as column vectors: $\mathbf{X} = [\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}]$
- Then the similar matrix $\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX}$ is diagonal with the eigenvalues of **A** as the diagonal elements

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \Rightarrow \quad \mathbf{D} = \mathbf{X}^{-1}\mathbf{AX} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- Companion relation: $\mathbf{XDX}^{-1} = \mathbf{A}$

Matrix Diagonalization Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \quad \lambda_1 = -5, \mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \lambda_2 = 2, \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \quad \mathbf{X}^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -5 & 4 \\ 15 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$$

3x3 Example

- Characteristic matrix

$$\mathbf{A} = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \mathbf{A} - \boldsymbol{\lambda}\mathbf{I} = \begin{bmatrix} \lambda + 6 & 11 & 6 \\ -1 & \lambda & 0 \\ 0 & -1 & \lambda \end{bmatrix}$$

- Characteristic equation

$$\left| \mathbf{A} - \boldsymbol{\lambda}\mathbf{I} \right| = (\lambda + 6)\lambda^2 + 6 + 11\lambda = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$\left| \mathbf{A} - \boldsymbol{\lambda}\mathbf{I} \right| = (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0 \quad \lambda_1 = -1 \quad \lambda_2 = -2 \quad \lambda_3 = -3$$

3x3 Example

- Eigenvector equation: $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$

$$\begin{aligned} -6x_1 - 11x_2 - 6x_3 &= \lambda x_1 \\ x_1 &= \lambda x_2 \\ x_2 &= \lambda x_3 \end{aligned} \quad\Rightarrow\quad \begin{aligned} -(6+\lambda)x_1 - 11x_2 - 6x_3 &= 0 \\ x_1 - \lambda x_2 &= 0 \\ x_2 - \lambda x_3 &= 0 \end{aligned}$$

- Eigenvector for $\lambda_1 = -1$

$$\begin{aligned} -5x_1 - 11x_2 - 6x_3 &= 0 \\ x_1 + x_2 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} 5x_2 - 11x_2 + 6x_2 &= 0 \\ x_1 &= -x_2 \\ x_3 &= -x_2 \end{aligned}$$

- Choose $x_2 = -1$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

3x3 Example

- Eigenvector for λ_2 = -2

$$\begin{aligned} -4x_1 - 11x_2 - 6x_3 &= 0 \\ x_1 + 2x_2 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 8x_2 - 11x_2 + 3x_2 &= 0 \\ x_1 &= -2x_2 \\ x_3 &= -\tfrac{1}{2}x_2 \end{aligned} \quad \Rightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

- Eigenvector for λ_3 = -3

$$\begin{aligned} -3x_1 - 11x_2 - 6x_3 &= 0 \\ x_1 + 3x_2 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 9x_2 - 11x_2 + 2x_2 &= 0 \\ x_1 &= -3x_2 \\ x_3 &= -\tfrac{1}{3}x_2 \end{aligned} \quad \Rightarrow \quad \mathbf{x}_3 = \begin{bmatrix} 9 \\ -3 \\ 1 \end{bmatrix}$$

3x3 Example

- Diagonalization

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 \\ -1 & -2 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$