Matrix Inverse

1. Existence and uniqueness
2. Determinants and matrix inverses
3. Gauss-Jordan elimination
4. In-class exercise

Matrix Inverse

Existence and Uniqueness

Eugène Rouché
1887

Alfredo Capelli
1887

Fundamental Theorem

- Linear algebraic system

$$\mathbf{Ax} = \mathbf{b} \quad \Rightarrow \quad \tilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{b}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

- Consistency
  - » The system has solutions if and only if the matrices $\mathbf{A}$ and $\tilde{\mathbf{A}}$ have the same rank r
- Uniqueness
  - » The system has a single solution if and only if both matrices have rank $r = n$
- Infinitely many solutions
  - » The system has infinitely many solutions if and only if both matrices have rank $r < n$

Implications

- Homogeneous system

$$\mathbf{Ax} = 0 \quad \Rightarrow \quad \tilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{b}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & 0 \end{bmatrix}$$

  » Trivial solution: $\mathbf{x} = \mathbf{0}$

  » Nontrivial solutions exist if and only if $rank(\mathbf{A}) < n$

  » Nontrivial solutions are said to be contained in the null space of **A**

- Nonhomogeneous system: $\mathbf{Ax} = \mathbf{b}$

  » If the system is consistent, then all solutions can be represented as $\mathbf{x} = \mathbf{x_0} + \mathbf{x_h}$

  » $\mathbf{x_0}$ is a particular solution of the nonhomogeneous system

  » $\mathbf{x_h}$ is any solution of the homogeneous system

Fundamental Theorem Examples

- Unique solution

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{matrix} rank(\mathbf{A}) = 2 \\ rank(\tilde{\mathbf{A}}) = 2 \end{matrix} \quad \mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Infinitely many solutions

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{matrix} rank(\mathbf{A}) = 1 \\ rank(\tilde{\mathbf{A}}) = 1 \end{matrix} \quad \mathbf{x} = \begin{bmatrix} 1 - 2a \\ a \end{bmatrix}$$

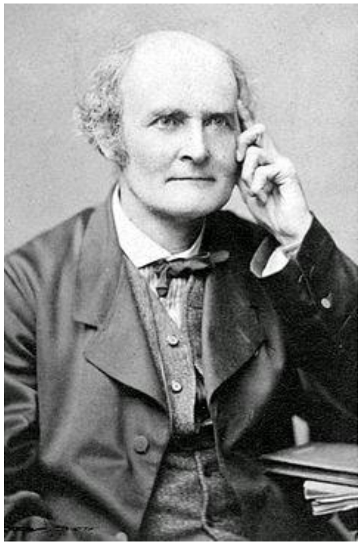
- No solutions

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{matrix} rank(\mathbf{A}) = 1 \\ rank(\tilde{\mathbf{A}}) = 2 \end{matrix}$$

No solutions exist

Matrix Inverse

Determinants and Matrix Inverse



Arthur Cayley
1855

Determinants

- Only applicable to square matrices
- Notation: det(**A**), |**A**|
- 2x2 matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- Example

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

Determinants

- 3x3 matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{matrix}$$

- Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{matrix} (1)(5)(9) + (2)(6)(7) + (3)(4)(8) \\ -(1)(6)(8) - (2)(4)(9) - (3)(5)(7) \end{matrix} = 0$$

- More general formulas based on cofactors are presented in the text

Properties of Determinants

- $|\mathbf{A}| = |\mathbf{A}^T|$
- Diagonal and triangular matrices

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}$$

- Products: $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$
- A zero column or row produces a zero determinant
- Linearly dependent rows or columns produce a zero determinant
- A square matrix **A** has full rank *n* if and only if $|\mathbf{A}|$ is non-zero

Matrix Inverse

- Definition
  - » Assume **A** is a *n*x*n* matrix
  - » The inverse of **A** is denoted $\mathbf{A}^{-1}$
  - » The inverse satisfies the equations: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
- Existence and uniqueness
  - » The inverse exists if and only if: $\det(\mathbf{A}) \neq 0$
  - » If **A** has an inverse, then the inverse is unique
- Equivalencies
  - » Singular matrix: $\mathbf{A}^{-1}$ does not exist, $\det(\mathbf{A}) = 0$, $\text{rank}(\mathbf{A}) < n$
  - » Nonsingular matrix: $\mathbf{A}^{-1}$ exists, $\det(\mathbf{A})$ non-zero, $\text{rank}(\mathbf{A}) = n$
- If $\text{rank}(\mathbf{A}) < n$, the matrix is said to rank deficient

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Special Cases

- 2x2 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- Diagonal matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^{-1} = \begin{bmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$$

- Product of square matrices

$$(\mathbf{AB}\cdots\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}\cdots\mathbf{B}^{-1}\mathbf{A}^{-1}$$

Matrix Inverse

Gauss-Jordan Elimination

Wilhelm Jordan
1888

Gauss-Jordan Elimination

- Involves transformation of the augmented matrix

$$[\mathbf{A} \quad \mathbf{I}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{bmatrix}$$

$$\Downarrow$$

$$[\mathbf{I} \quad \mathbf{A}^{-1}] = \begin{bmatrix} 1 & \cdots & 0 & \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}$$

- Only row operations are allowed
- Operations on columns are not allowed because only the rows represent equations

Gauss-Jordan Elimination Example

- Method to compute $\mathbf{A}^{-1}$ using row operations
- Form augmented matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \quad \Rightarrow \quad \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

- Eliminate first entry in last two rows

$$\begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 3+3(-1) & -1+3(1) & 1+3(2) & 0+3(1) & 1+3(0) & 0+3(0) \\ -1+1 & 3-1 & 4-2 & 0-1 & 0-0 & 1-0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{bmatrix}$$

Gauss-Jordan Elimination Example

- Eliminate x_2 entry from third row

$$\begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0-0 & 2-2 & 2-7 & -1-3 & 0-1 & 1-0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{bmatrix}$$

- Make the diagonal elements unity

$$\begin{bmatrix} -(-1) & -1 & -2 & -1 & 0 & 0 \\ 0 & 0.5(2) & 0.5(7) & 0.5(3) & 0.5(1) & 0 \\ 0 & 0 & -0.2(-5) & -0.2(-4) & -0.2(-1) & -0.2(1) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Gauss-Jordan Elimination Example

- Eliminate first two entries in third column

$$\begin{bmatrix} 1+2(0) & -1+2(0) & -2+2(1) & -1+2(0.8) & 0+2(0.2) & 0+2(-0.2) \\ 0 & 1-3.5(0) & 3.5-3.5(1) & 1.5-3.5(0.8) & 0.5-3.5(0.2) & 0-3.5(-0.2) \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix}$$

- Obtain identity matrix

$$\begin{bmatrix} 1 & -1+1 & 0 & 0.6+(-1.3) & 0.4+(-0.2) & -0.4+0.7 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}^{-1} \end{bmatrix}$$

- Matrix inverse

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Gauss-Jordan Elimination Example

$$\mathbf{C} = \mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$c_{11} = (-1)(-0.7) + (1)(-1.3) + (2)(0.8) = 1$$

$$c_{12} = (-1)(0.2) + (1)(-0.2) + (2)(0.2) = 0$$

$$c_{13} = (-1)(0.3) + (1)(0.7) + (2)(-0.2) = 0$$

$$\vdots$$

$$\mathbf{C} = \mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

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Using the Matrix Inverse

- Linear algebraic equation system: $\mathbf{Ax} = \mathbf{b}$
  - » Assume **A** is a non-singular matrix
  - » Solution

$$\mathbf{Ax} = \mathbf{b} \quad \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \quad \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

- Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Matrix Inverse

In-class Exercise