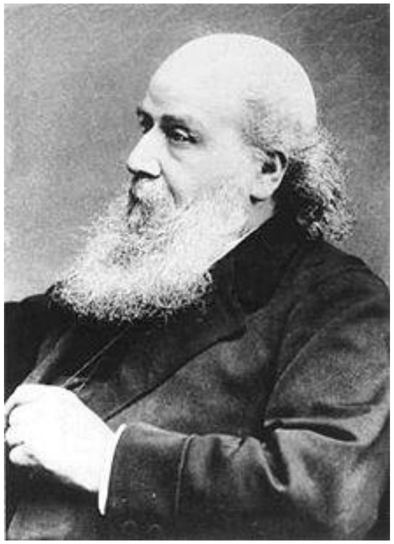
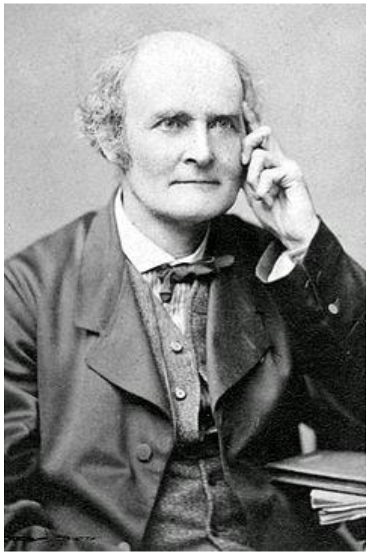
# Linear Algebra

1. Basic concepts
2. Matrix operations
3. In-class exercise
4. Linear independence and matrix rank

**Hermann Grassmann**
**1844**



**James Sylvester**
**1848**



**Arthur Cayley**
**1858**

# Linear Algebra

Basic Concepts

# Motivating Example

## Reaction Sequence

| | |
|---|---|
| Reaction in ($v_i$) | A’ → A |
| Reaction 1 ($v_1$) | A+2F → 2B+2F’ |
| Reaction 2 ($v_2$) | B+E’ → C+E |
| Reaction 3 ($v_3$) | C+2F’ → D+2F |
| Reaction 4 ($v_4$) | D+E → G+E’ |
| Reaction 5 ($v_5$) | F → F’ |
| Reaction 6 ($v_6$) | B+E → H+E’ |
| Reaction out ($v_o$) | D → D’ |

## Component Balances

$$\mathrm{A}:\ 0 = v_i - v_1 \qquad \mathrm{B}:\ 0 = 2v_1 - v_2 - v_6$$

$$\mathrm{C}:\ 0 = v_2 - v_3 \qquad \mathrm{D}:\ 0 = v_3 - v_4 - v_0$$

$$\mathrm{E}:\ 0 = v_2 - v_4 - v_6 \qquad \mathrm{F}:\ 0 = -2v_1 + 2v_3 - v_5$$

# Motivating Example

- Component balances on E' and F' are redundant with those for E and F:

$$E': \; 0 = -v_2 + v_4 + v_6 \qquad F': \; 0 = 2v_1 - 2v_3 + v_5$$

- Objective is to solve these 6 equations for the 6 unknowns
- Manipulating the scalar equations is tedious and inefficient
- We seek a more effective method to solve systems of linear algebraic equations of the form:

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n
\end{gathered}$$

# Basic Notation

- $m$-dimensional column vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

- $n$-dimensional row vector

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} 4 & 2 & -5 \end{bmatrix}$$

- $m$x$n$-dimensional matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & -6 \\ 2 & 0 \end{bmatrix}$$

- Square matrix: $m = n$

# Linear Algebra

Matrix Operations

# Matrix Addition and Subtraction

- Only defined for matrices of same dimension
- Add/subtract matrices element-by-element
- Addition example: **C** = **A**+**B**

$$\begin{bmatrix} 1 & 3 & -2 \\ 4 & 2 & 3 \\ -1 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & -1 \\ 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 5 & -3 \\ 7 & 3 & 5 \\ 3 & 8 & 2 \end{bmatrix}$$

- Subtraction example: **C** = **A**-**B**

$$\begin{bmatrix} 4 & 2 & -1 \\ 5 & 3 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 2 \\ -4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -3 \\ 9 & 0 & -6 \end{bmatrix}$$

# Scalar and Matrix Multiplication

- Scalar multiplication
  - » $\mathbf{B} = k\mathbf{A}$
  - » Dimensions: $k \in R \quad \mathbf{A} \in R^{mxn} \quad \mathbf{B} \in R^{mxn}$
  - » General formula: $b_{ij} = ka_{ij} \quad B = \{b_{ij}\}$
  - » Example

$$3\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -3 & 3 \end{bmatrix}$$

- Matrix multiplication
  - » $\mathbf{C} = \mathbf{AB}$
  - » Only possible if the number of columns of **A** is equal to the number of rows of **B**

# Matrix Multiplication cont.

- General representation
  - » Dimensions: $\mathbf{A} \in R^{mxn}$ $\quad \mathbf{B} \in R^{nxr}$ $\quad \mathbf{C} \in R^{mxr}$
  - » Formula

$$c_{jk} = \sum_{i=1}^{n} a_{ji} b_{ik} \qquad j = 1,\ldots,m \quad k = 1,\ldots,r$$

- Examples

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -4 & -8 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = 3$$

- Noncommutative operation: $\mathbf{AB} \neq \mathbf{BA}$

# Motivating Example Revisited

- Component balances in scalar form

$$\mathrm{A}:\ 0 = v_i - v_1 \qquad \mathrm{B}:\ 0 = 2v_1 - v_2 - v_6$$

$$\mathrm{C}:\ 0 = v_2 - v_3 \qquad \mathrm{D}:\ 0 = v_3 - v_4 - v_0$$

$$\mathrm{E}:\ 0 = v_2 - v_4 - v_6 \qquad \mathrm{F}:\ 0 = -2v_1 + 2v_3 - v_5$$

- Component balances in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ -2 & 0 & 2 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} v_i \\ 0 \\ 0 \\ v_o \\ 0 \\ 0 \end{bmatrix}$$

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# Transpose

- Notation: $\mathbf{B} = \mathbf{A}^{\mathrm{T}}$
- Dimensions: $\mathbf{A} \in R^{mxn}$ $\quad \mathbf{B} \in R^{nxm}$
- Formula: $b_{ij} = a_{ji} \quad B = \{b_{ij}\}$
- Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

- Useful properties

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(c\mathbf{A})^T = c\mathbf{A}^T$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

# Linear and Quadratic Forms

- Linear form

$$\mathbf{c}^T\mathbf{x} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

- Quadratic form

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1x_1 & d_2x_2 & \cdots & d_nx_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{x}^T\mathbf{D}\mathbf{x} = d_1x_1^2 + d_2x_2^2 + \cdots + d_nx_n^2$$

# Common Matrices

- Symmetric matrix: $\mathbf{A}^{T} = \mathbf{A}$
- Skew-symmetric matrix: $\mathbf{A}^{T} = -\mathbf{A}$
- Diagonal matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

- Triangular matrices

$$\text{Upper}: \quad \mathbf{A} = \begin{bmatrix} 5 & 3 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{Lower}: \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$$

# Common Matrices

- Positive definite matrix – for any non-zero vector **x**:

$$\mathbf{x}^T \mathbf{A}\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} > 0$$

- Identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

# Linear Algebra

In-class Exercise

# Linear Algebra

Linear Independence and Matrix Rank

# Linear Independence

- Given $m$ vectors $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$, …, $\mathbf{a}_{(m)}$ of equal dimension
- Consider the linear equation

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = 0$$

- Linear independent vectors
  - » Equation satisfied only for $c_j = 0$
  - » Each vector is “unique”
- Linear dependent vectors
  - » Equation also satisfied for some non-zero $c_j$
  - » At least one vector is “redundant”

# Linear Independence

- Example of linearly independent vectors

$$\begin{aligned} \mathbf{a}_{(1)} &= [1 \quad 2 \quad 3] \\ \mathbf{a}_{(2)} &= [2 \quad -3 \quad 1] \quad \Rightarrow \quad 0\mathbf{a}_{(1)} + 0\mathbf{a}_{(2)} + 0\mathbf{a}_{(3)} = \mathbf{0} \\ \mathbf{a}_{(3)} &= [4 \quad 1 \quad 8] \end{aligned}$$

- Example of linearly dependent vectors

$$\begin{aligned} \mathbf{a}_{(1)} &= [1 \quad 2 \quad 3] \\ \mathbf{a}_{(2)} &= [2 \quad -3 \quad 1] \quad \Rightarrow \quad 2\mathbf{a}_{(1)} + \mathbf{a}_{(2)} - \mathbf{a}_{(3)} = \mathbf{0} \\ \mathbf{a}_{(3)} &= [4 \quad 1 \quad 7] \end{aligned}$$

# Matrix Rank

- $r = rank(\mathbf{A})$
  - » Number of linearly independent row vectors of **A**
  - » Number of linearly independent column vectors of **A**
- Examples

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 4 & 1 & 7 \end{bmatrix} \quad rank(\mathbf{A}) = 2 \qquad \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 4 & 1 & 8 \end{bmatrix} \quad rank(\mathbf{A}) = 3$$

- Rank can be determined through elementary row operations (see next lecture)
- Square matrix must be full rank for linear algebraic system to yield a unique solution (see next lecture)