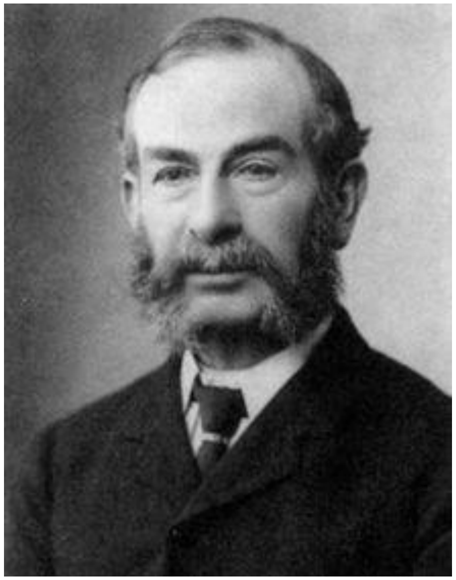
Introduction to ODE Stability

1. Linear ODE system stability
2. Nonlinear ODE system stability
3. Biochemical reactor model
4. In-class exercise

Adolf Hurwitz
1895



Edward Routh
1895

Introduction to ODE Stability

Linear ODE System Stability

Introduction

- Stability is a concept that only applies to time dependent ODEs
- Basic idea
  - » Consider a nonlinear ODE system with the origin as a steady-state point:

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) \quad \Rightarrow \quad \mathbf{f}(\mathbf{0}) = \mathbf{0}$$

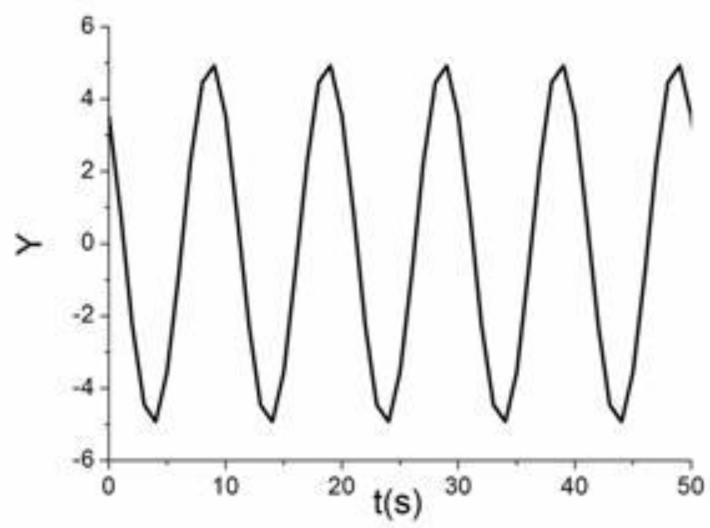
  - » Does the system return to the origin if perturbed away from the origin? If so, the system is stable. Otherwise, the system is unstable.

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}) \quad \|\mathbf{y}(0)\| = \varepsilon \quad \Rightarrow \quad \lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$$

- Linear system stability is completely determined by the eigenvalues

Different Types of Stability

- Asymptotic stability
  - » System converges to a point
  - » Precludes limit cycle solutions
- Stability
  - » System remains bounded
  - » Allows limit cycle solutions



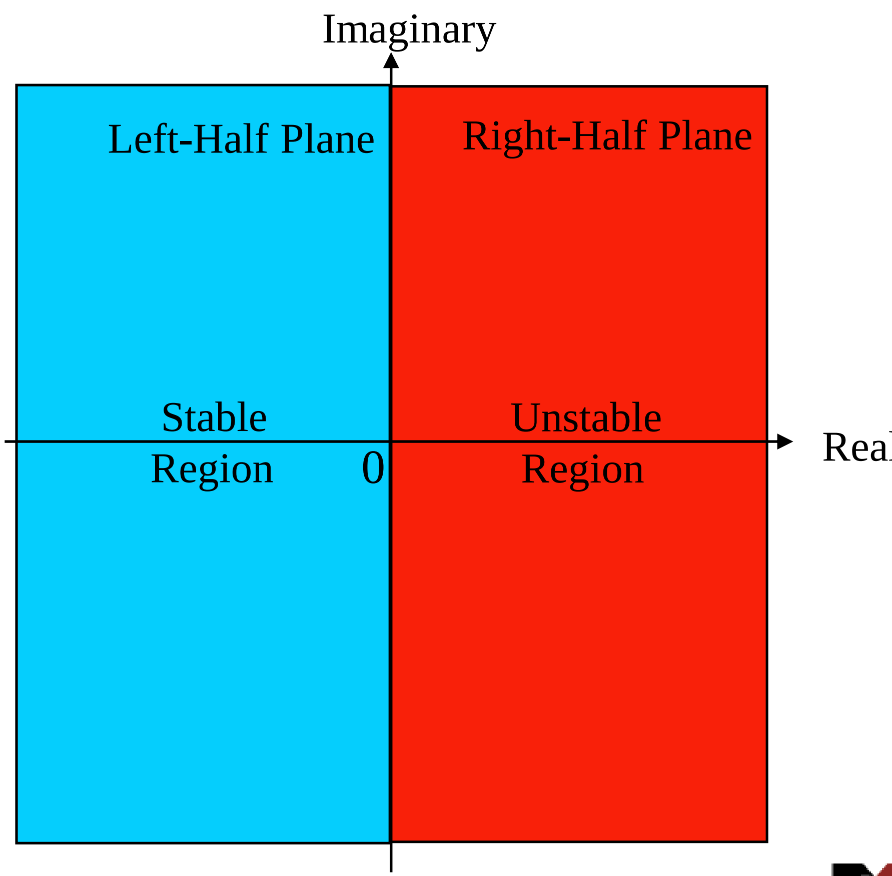
- Global stability
  - » Stability guaranteed for any initial condition
  - » Strongest form of stability
- Local stability
  - » Stability guaranteed only in some vicinity of the initial condition
  - » Typical form of nonlinear system stability

Linear Stability Analysis

- General solution form for distinct eigenvalues:

$$\mathbf{y}(t) = c_1 \mathbf{x}^{(1)} e^{\lambda_1 t} + c_2 \mathbf{x}^{(2)} e^{\lambda_2 t} + \cdots + c_n \mathbf{x}^{(n)} e^{\lambda_n t}$$

- Analysis procedure
  - » Compute the eigenvalues of **A**
  - » The system is asymptotically stable if and only if Re(λ_i) < 0 for i = 1, 2, ..., n
  - » The origin is unstable if Re(λ_i) > 0 for any i
  - » Stability allows zero eigenvalues



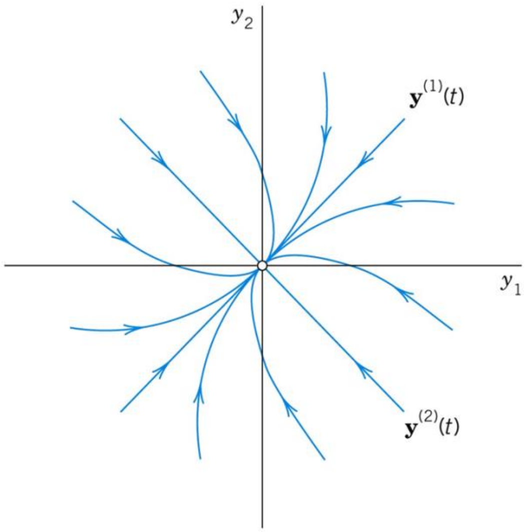
Two-Dimensional Linear Systems

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} \quad \Rightarrow \quad \lambda(\mathbf{A}) = \lambda_1, \lambda_2$$

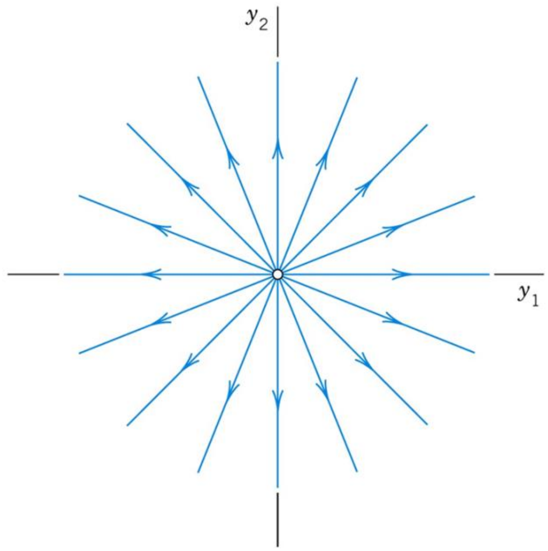
- The qualitative nature of the solution is determined by the eigenvalues
  - » Improper node – real, distinct eigenvalues
  - » Proper node – real, repeated eigenvalues
  - » Saddle point – real eigenvalues with different signs
  - » Center – imaginary eigenvalues
  - » Spiral – complex eigenvalues
  - » Degenerate node – no eigenvector basis (rare)
- Asymptotic stability still requires eigenvalues to have negative real parts

Phase Planes

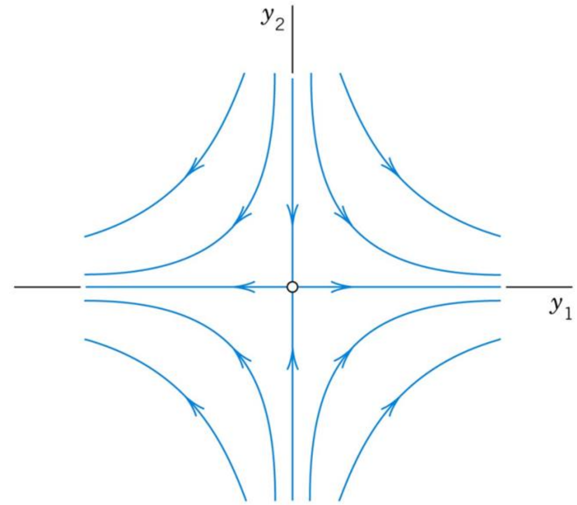
Improper node



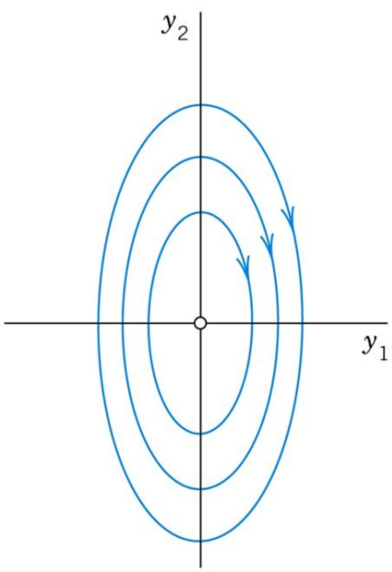
Proper node



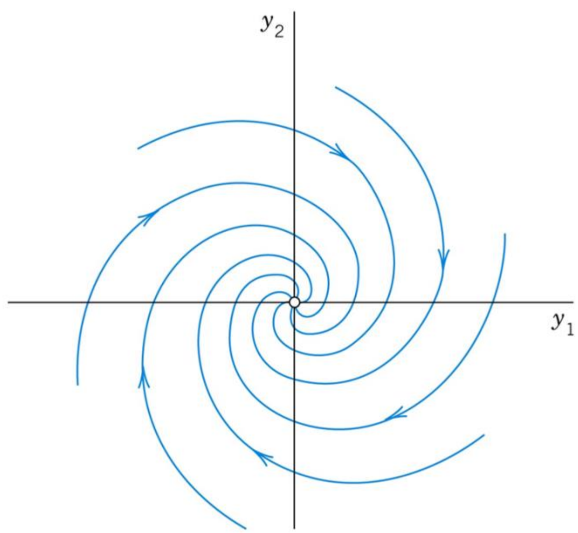
Saddle point



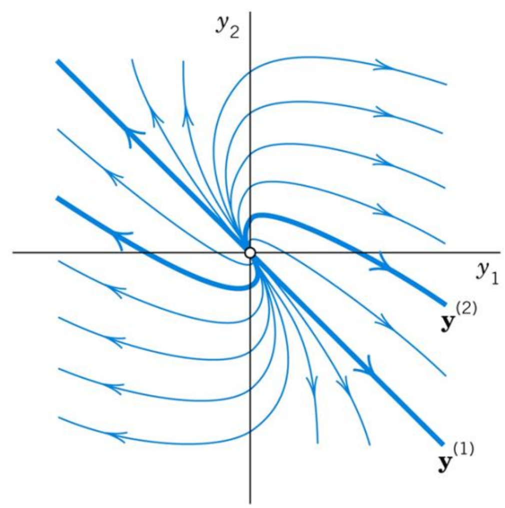
Center



Spiral



Degenerate node



Introduction to ODE Stability

Nonlinear ODE System Stability

Linearization of Nonlinear Systems

- Linear stability analysis can be extended to nonlinear systems through linearization
- First the steady-state solutions of the nonlinear system are determined
- Second the nonlinear system is linearized about a steady state to generate a linearized system
- Third the stability of the original nonlinear system is deduced from the eigenvalues of the linearized system
- The analysis is local because the linearized system is only a "good" approximation of the original nonlinear system "near" the steady state
- The procedure must be repeated at each steady state of interest

Linearization of One-Dimensional System

- Nonlinear ODE model

$$\frac{dy}{dt} = f(y) \qquad f(\bar{y}) = 0$$

- First-order Taylor series expansion about steady state

$$\frac{dy}{dt} \cong f(\bar{y}) + \left(\frac{\partial f}{\partial y}\right)_{(\bar{y})}(y - \bar{y}) = \left(\frac{\partial f}{\partial y}\right)_{(\bar{y})}(y - \bar{y})$$

- Generate linear ODE model

$$y'(t) = y(t) - \bar{y} \qquad \frac{dy'}{dt} \cong \left(\frac{\partial f}{\partial y}\right)_{(\bar{y})} y' = ay'$$

Chemical Reactor Example

- Nonlinear ODE model

$$2A \xrightarrow{k} B \qquad \frac{dC_A}{dt} = \frac{q}{V}(C_{Af} - C_A) - 2kC_A^2 = f(C_A)$$

- Find steady-state point (q = 2, V = 2, C_{Af} = 2, k = 0.5)

$$f(\overline{C}_A) = \frac{q}{V}(C_{Af} - \overline{C}_A) - 2k\overline{C}_A^2 = \frac{2}{2}(2 - \overline{C}_A) - (2)(0.5)\overline{C}_A^2 = 0$$

$$\overline{C}_A^2 + \overline{C}_A - 2 = 0$$

$$\overline{C}_A = \frac{-1 \pm \sqrt{1^2 - (4)(1)(-2)}}{(2)(1)} = \frac{-1 \pm 3}{2} = 1, -2$$

Chemical Reactor Example

- Linearize about steady-state point:

$$\frac{dC_A^{'}}{dt} \cong \underbrace{f(\overline{C}_A)}_{0} + \left(\frac{\partial f}{\partial C_A}\right)_{\overline{C}_A} C_A^{'}$$

$$\frac{dC_A^{'}}{dt} \cong [-\overline{C}_A - (2)(2)(0.5)\overline{C}_A]C_A^{'} = -3C_A^{'}$$

Two-Dimensional System

- Nonlinear ODE model

$$\frac{dy_1}{dt} = f_1(y_1, y_2) \quad \Rightarrow \quad f_1(\bar{y}_1, \bar{y}_2) = 0$$
$$\frac{dy_2}{dt} = f_2(y_1, y_2) \quad \Rightarrow \quad f_2(\bar{y}_1, \bar{y}_2) = 0$$

- First-order Taylor series expansion

$$\frac{dy_1}{dt} \cong \underbrace{f_1(\bar{y}_1, \bar{y}_2)}_{0} + \left(\frac{\partial f_1}{\partial y_1}\right)_{(\bar{\mathbf{y}})} (y_1 - \bar{y}_1) + \left(\frac{\partial f_1}{\partial y_2}\right)_{(\bar{\mathbf{y}})} (y_2 - \bar{y}_2)$$
$$\frac{dy_2}{dt} \cong \underbrace{f_2(\bar{y}_1, \bar{y}_2)}_{0} + \left(\frac{\partial f_2}{\partial y_1}\right)_{(\bar{\mathbf{y}})} (y_1 - \bar{y}_1) + \left(\frac{\partial f_2}{\partial y_2}\right)_{(\bar{\mathbf{y}})} (y_2 - \bar{y}_2)$$

Two-Dimensional System

- Linearized ODE model

$$\frac{dy_1'}{dt} = a_{11}y_1' + a_{12}y_2'$$
$$\frac{dy_2'}{dt} = a_{21}y_1' + a_{22}y_2'$$
$$\Rightarrow \quad \frac{d\mathbf{y}'}{dt} = \mathbf{A}\mathbf{y}' = \mathbf{J}(\bar{\mathbf{y}})\mathbf{y}' \quad \mathbf{y}'(0) = \mathbf{y}_0 - \bar{\mathbf{y}}$$

Linearized Stability Analysis

- Procedure
  - » Linearize the nonlinear model at a steady state to determine the **A** matrix
  - » Compute the eigenvalues of **A**
  - » The steady state is locally asymptotically stable if $\mathrm{Re}(\lambda_i) < 0$ for i = 1, 2, ..., n
  - » The steady state is unstable if $\mathrm{Re}(\lambda_i) > 0$ for any i
  - » More advanced methods needed if $\mathrm{Re}(\lambda_i) = 0$
- Comments
  - » Nonlinear systems may have more than one stable state
  - » Both steady states and limit cycle solutions can be stable
  - » Each stable state has a certain domain of attraction with respect to the initial conditions

Introduction to ODE Stability

Biochemical Reactor Model

Biochemical Reactor Model

- Model equations

$$\frac{dX}{dt} = -DX + \mu(S)X = f_1(X,S) \qquad \mu(S) = \frac{\mu_m S}{K_S + S}$$

$$\frac{dS}{dt} = D(S_0 - S) - \frac{1}{Y_{X/S}}\mu(S)X = f_2(X,S)$$

- Steady-state equations

$$-D\bar{X} + \mu(\bar{S})\bar{X} = 0 \qquad \mu(\bar{S}) = \frac{\mu_m \bar{S}}{K_S + \bar{S}}$$

$$D(S_0 - \bar{S}) - \frac{1}{Y_{X/S}}\mu(\bar{S})\bar{X} = 0$$

- Two steady-state points

$$\text{Non-Trivial}: \quad \mu(\bar{S}) = D \Rightarrow \bar{S} = \frac{K_S D}{\mu_m - D} \qquad \bar{X} = Y_{X/S}(S_0 - \bar{S})$$

$$\text{Washout}: \qquad \bar{S} = S_0 \qquad \bar{X} = 0$$

Biochemical Reactor Model

- Linearize the biomass concentration equation

$$\frac{dX'}{dt} \cong \underbrace{f_1(\bar{X},\bar{S})}_{\text{zero}} + \left[\frac{\partial f_1}{\partial X}\right]_{\bar{X},\bar{S}} \left(X-\bar{X}\right) + \left[\frac{\partial f_1}{\partial S}\right]_{\bar{X},\bar{S}} \left(S-\bar{S}\right)$$

$$= \left[\mu(\bar{S}) - D\right]X' + \left[\frac{\mu_m \bar{X}}{K_S+\bar{S}} - \frac{\mu_m \bar{X}\bar{S}}{\left(K_S+\bar{S}\right)^2}\right]S'$$

- Linearize the substrate concentration equation

$$\frac{dS'}{dt} \cong \underbrace{f_2(\bar{X},\bar{S})}_{\text{zero}} + \left[\frac{\partial f_2}{\partial X}\right]_{\bar{X},\bar{S}} \left(X-\bar{X}\right) + \left[\frac{\partial f_2}{\partial S}\right]_{\bar{X},\bar{S}} \left(S-\bar{S}\right)$$

$$= -\frac{1}{Y_{X/S}} \frac{\mu_m \bar{S}}{K_S+\bar{S}} X' - \left[\frac{1}{Y_{X/S}}\left(\frac{\mu_m \bar{X}}{K_S+\bar{S}} - \frac{\mu_m \bar{X}\bar{S}}{\left(K_S+\bar{S}\right)^2}\right) + D\right]S'$$

Biochemical Reactor Model

- Parameter values
  - » K_S = 1.2 g/L, μ_m = 0.48 h^{-1}, $Y_{X/S}$ = 0.4 g/g
  - » D = 0.15 h^{-1}, S_0 = 20 g/L
- Non-trivial steady state

$$\bar{S} = \frac{K_S D}{\mu_m - D} = 0.545\ \text{g/L} \qquad \bar{X} = Y_{X/S}(S_0 - \bar{S}) = 7.78\ \text{g/L}$$

- Linearized model

$$\frac{dX'}{dt} = a_{11}X' + a_{12}S'$$

$$\frac{dS'}{dt} = a_{21}X' + a_{22}S'$$

$$a_{11} = 0 \qquad a_{12} = \frac{\mu_m \bar{X}}{K_S + \bar{S}} - \frac{\mu_m \bar{X}\bar{S}}{\left(K_S + \bar{S}\right)^2} = 1.472$$

$$a_{21} = -\frac{1}{Y_{X/S}} \frac{\mu_m \bar{S}}{K_S + \bar{S}} = -0.375 \qquad a_{22} = -\frac{1}{Y_{X/S}}\left(\frac{\mu_m \bar{X}}{K_S + \bar{S}} - \frac{\mu_m \bar{X}\bar{S}}{\left(K_S + \bar{S}\right)^2}\right) + D = -3.529$$

Biochemical Reactor Model

- Matrix representation

$$\mathbf{y} = \begin{bmatrix} X' \\ S' \end{bmatrix} \Rightarrow \frac{d\mathbf{y}}{dt} = \begin{bmatrix} 0 & 1.472 \\ -0.375 & -3.529 \end{bmatrix} \mathbf{y} = \mathbf{A}\mathbf{y}$$

- Eigenvalues

$$|\mathbf{A} - \lambda I| = \begin{vmatrix} -\lambda & 1.472 \\ -0.375 & -3.529 - \lambda \end{vmatrix} \Rightarrow \lambda_1 = -0.164 \quad \lambda_2 = -3.365$$

- Conclusion
  - » Non-trivial steady state is asymptotically stable
  - » Result holds locally near the steady state

Biochemical Reactor Model

- Washout steady state: $\bar{S} = S_i = 20\,\text{g/L} \quad \bar{X} = 0\,\text{g/L}$
- Linearized model coefficients

$$a_{11} = \frac{\mu^{\max}\bar{S}}{K_S + \bar{S}} - D = 0.303 \qquad a_{12} = 0$$

$$a_{21} = -\frac{1}{Y_{X/S}}\frac{\mu^{\max}\bar{S}}{K_S + \bar{S}} = -1.132 \qquad a_{22} = -\frac{1}{Y_{X/S}}\left(\frac{\mu^{\max}\bar{X}}{K_S + \bar{S}} - \frac{\mu^{\max}\bar{X}\bar{S}}{\left(K_S + \bar{S}\right)^2}\right) + D = 0.15$$

- Eigenvalues

$$|A - \lambda I| = \begin{vmatrix} 0.303 - \lambda & 0 \\ -1.132 & 0.15 - \lambda \end{vmatrix} \quad \Rightarrow \quad \lambda_1 = +0.303 \quad \lambda_1 = +0.15$$

- Conclusion
  - » Washout steady state is unstable
  - » Suggests that non-trivial steady state is globally stable

Introduction to ODE Stability

In-class Exercise