Gauss Elimination

1. Gauss elimination method
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3. In-class exercise
4. Matrix rank and pivoting

Carl Friedrich Gauss
1810

Gauss Elimination

Gauss Elimination Method

Square Linear Algebraic Systems

- Scalar representation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
\begin{aligned}
2x_1 - 3x_2 &= 3 \\
-x_1 - 2x_2 &= 6
\end{aligned}
$$

- Matrix representation: $\mathbf{Ax} = \mathbf{b}$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

- Homogeneous system: $\mathbf{b} = \mathbf{0}$
  - » Trivial solution: $\mathbf{x} = \mathbf{0}$
  - » Seek non-trivial solutions

Triangular Systems

- Example

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

- Solution

$$2x_1 = 4 \Rightarrow x_1 = 2$$

$$x_1 - x_2 = 3 \Rightarrow x_2 = x_1 - 3 = -1$$

$$-2x_1 - 3x_2 + 2x_3 = 1 \Rightarrow x_3 = \frac{1 + 2x_1 + 3x_2}{2} = 1$$

- Gauss elimination
  - » Transform original system into diagonal form
  - » Accomplished by elementary row operations

Gauss Elimination

- Augmented matrix

$$\mathbf{Ax} = \mathbf{b} \quad \Rightarrow \quad \tilde{\mathbf{A}} = [\mathbf{A} \quad \mathbf{b}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{bmatrix}$$

- Elementary row operations
  - » Interchange of two rows
  - » Multiplication of a row by a non-zero constant
  - » Addition of a constant multiple of one row to another row
  - » Operations on columns are not allowed because only the rows represent equations
- Perform row operations until augmented system becomes triangular

Gauss Elimination

Gauss Elimination Examples

Gauss Elimination Example #1

- Form augmented matrix

$$\begin{bmatrix} 3 & -2 & 2 \\ -5 & 4 & -3 \\ -4 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \tilde{\mathbf{A}} = \begin{bmatrix} 3 & -2 & 2 & -1 \\ -5 & 4 & -3 & 3 \\ -4 & 3 & -2 & 1 \end{bmatrix}$$

- Eliminate x_1 from second and third equations

$$\begin{bmatrix} 3 & -2 & 2 & -1 \\ 0 & 4-10/3 & -3+10/3 & 3-5/3 \\ 0 & 3-8/3 & -2+8/3 & 1-4/3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 & -1 \\ 0 & 2/3 & 1/3 & 4/3 \\ 0 & 1/3 & 2/3 & -1/3 \end{bmatrix}$$

Gauss Elimination Example #1

- Eliminate x_2 from third equation

$$\begin{bmatrix} 3 & -2 & 2 & -1 \\ 0 & 2/3 & 1/3 & 4/3 \\ 0 & 0 & 2/3-1/6 & -1/3-4/6 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 & -1 \\ 0 & 2/3 & 1/3 & 4/3 \\ 0 & 0 & 1/2 & -1 \end{bmatrix}$$

- Solve triangular system

$$\begin{bmatrix} 3 & -2 & 2 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4/3 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

- Solution is unique

Gauss Elimination Example #2

- Form augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

- Eliminate x_1 from second and third equations

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Equations:

$$x_1 + x_2 - x_3 = -1, \quad x_3 = 1$$

- Infinite number of solutions

Gauss Elimination Example #3

- Form augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

- Eliminate x_1 from second and third equations

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- Equations:

$$x_1 + x_2 - x_3 = -1, \quad 0 = -1, \quad x_3 = 1$$

- No solution exists

Gauss Elimination

In-class Exercise

Gauss Elimination

Matrix Rank and Pivoting

Matrix Rank

- After Gauss elimination the augmented matrix [**A**|**b**] can be represented in row echelon form [**R**|**f**]:

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1r} & \cdots & r_{1n} & f_1 \\ 0 & r_{22} & \cdots & r_{2r} & \cdots & r_{2n} & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{rr} & \cdots & r_{rn} & f_r \\ 0 & 0 & \cdots & 0 & \cdots & 0 & f_{r+1} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & f_n \end{bmatrix}$$

- The rank r of the matrix **A** is equal to the number of non-zero rows of **R**

Matrix Rank

$$\mathbf{Ax} = \mathbf{b}$$

- No solution exists if $r < n$ and at least one number $\{f_{r+1},\ldots,f_n\}$ is non-zero. The system is called inconsistent
- The system is called consistent and solutions exist if:
  - » $r = n$ or
  - » $r < n$ and all the numbers $\{f_{r+1},\ldots,f_n\}$ are zero
- Example 1: $r = 3 = n$ (unique solution)
- Example 2: $r = 2 < n = 3$ and $f_3 = 0$ (infinite number of solutions)
- Example 3: $r = 2 < n = 3$ and $f_3 = -1$ (no solutions)

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Gauss Elimination Example #4

- Form augmented matrix

$$\begin{bmatrix} 0 & 8 & 2 \\ 3 & 5 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 8 \\ 26 \end{bmatrix} \quad \Rightarrow \quad \tilde{\mathbf{A}} = \begin{bmatrix} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{bmatrix}$$

- Exchange equations 1 and 3 to obtain the largest possible non-zero <u>pivot</u> a_{11}. Multiple the <u>pivot equation</u> by -0.5 and add to the second equation.

$$\begin{bmatrix} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{bmatrix}$$

Gauss Elimination Example #4

- Exchange equations 2 and 3 to obtain the largest possible non-zero pivot a_{22}. Multiple the pivot equation by -0.5 and add to the third equation.

$$\begin{bmatrix} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -1.5 \end{bmatrix}$$

- Solve triangular system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0.5 \end{bmatrix}$$

- Small pivots can cause numerical problems

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Gauss Elimination Example #5

- Form the augmented matrix:

$$\begin{bmatrix} 0.0004 & 1.402 \\ 0.4003 & -1.502 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.406 \\ 2.501 \end{bmatrix} \quad \Rightarrow \quad \tilde{\mathbf{A}} = \begin{bmatrix} 0.0004 & 1.402 & 1.406 \\ 0.4003 & -1.502 & 2.501 \end{bmatrix}$$

- Do not pivot. Instead multiple the first equation by -0.4003/0.004 = 1001 and add to the second equation using 4 significant digits.

$$\begin{bmatrix} 0.0004 & 1.402 & 1.406 \\ 0 & -1405 & -1404 \end{bmatrix}$$

- Solution of triangular system not equal to true solution

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12.5 \\ 0.9993 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

Computational Efficiency

- Gauss elimination requires two steps
  - » Forward elimination to form a triangular system
  - » Back substitution to solve the triangular system
- To solve a *nxn* system of equations the number of operations *f* scales as:
  - » Elimination: $f(n) = O(n^3)$
  - » Substitution: $f(n) = O(n^2)$
- If each operation requires 10^{-9} seconds:

Step	**n = 1000**	**n = 10000**
Elimination	~1 seconds	~10 minutes
Substitution	~0.001 seconds	~0.1 seconds

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