A FEMLAB Study:
The Leveque Solution With a Finite Wall Resistance

by

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Introduction

The effect of heating a fluid at low Reynolds number flowing upward in a vertical tube has been studied experimentally and analytically [1,2,3,4]. The buoyancy force resulting from the heating may cause flow reversal and instabilities in the flow field [3].

Recently, Rosen [5] explored the use of FEMLAB [6, 7] to predict the point of flow reversal and compared it to experiment. In that study it was found that the mesh size in FEMLAB was an important factor in accurately determining the flow field in the entrance area where a step change in the wall temperature takes place. Computer memory limitations restricted the use of a finer mesh to describe the entire flow field.

It is the purpose of this study to compare a FEMLAB solution utilizing a short tube to an analytical solution (Leveque [8]) in the entrance region. In the very short entrance region buoyancy effects are assumed not to affect the FEMLAB solution.

Equations Describing the Problem

Fig (1) describes the system as simulated in FEMLAB [5]. The boundary condition for the conductive flux at the tube wall was specified as:

$$q_w/A = k*(\partial t/\partial r)_{r=a} = k_s * (t_d - t_w)/(a \ln (d/a)) \qquad (1)$$

where k and k_s are the thermal conductivity of the fluid (water) and tube wall respectively. The values of a and d are 0.01095 m, 0.01250 m respectively and k_s = 1.12508 J/(m-s-K).

Fig (2) is the FEMLAB results for the conductive flux at the entrance when a “regular” mesh size was used for Run 24 [5]. By regular is meant the that mesh size was initialized (using the initial method in FEMLAB) and then refined twice. When a tube length of about 0.6 m was used (radius 0.01095 m) adequate memory was available for the mesh generated. Note that in Fig (2) the conductive flux is zero across most of the radius (since the temperature at the entrance is constant) but at the wall the discontinuity (due to the start of the heated wall) causes an uneven and then a steep rise in the conductive flux.

The FEMLAB simulation for Run 24 described in [5] (t_o = 300.98 K, t_d = 315.09 K) was rerun with a very short tube (0.002 m). This allowed a fine mesh for which adequate computer memory was available (512 Mb). The thin tube wall and shortness of the tube permit an approximation as a flat plate to be made.

Leveque Solution With a Finite Wall Resistance

An analytical solution for the heat transfer from a flat plate with thermal resistance to a fluid is described by Rosen and Scott [8].

The wall temperature (t_w) is given in terms of the outside wall temperature (t_d) and the entering temperature of the fluid (t_o) as:

$$\frac{1}{1-T_d} = \frac{t_w - t_o}{t_d - t_o} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\eta Z^{*1/3})^n}{\Gamma(1+n/3)} \qquad (2)$$

where Γ (x) is the gamma function.

The conductive heat flux at the wall (plate) is described by:

$$\frac{q_w}{A} = \frac{k\,\lambda\,(t_d - t_o)}{a} \left[1 + \sum_{n=1}^{\infty} (-1)^{n} \frac{(\eta Z^{*1/3})^n}{\Gamma(1+n/3)} \right] \qquad (3)$$

where λ is a parameter equal to k_s/(k ln (d/a).

Spreadsheet Development

The spreadsheet of Fig (3) was designed to

1. Record the parameters of Run 24 taken from [3,5] . These include the values Re, Pr, a, t_d, t_o , k and k_s. The physical properties were evaluated at the entering fluid’s temperature, t_o.

2. Record the results of the FEMLAB simulation for Run 24 with a tube length of 0.002 m. The temperature and and conductive wall flux values were determined by reading point values from FEMLAB post processing graphics.

3. Evaluate Equations (2) and (3) and record the results for the same values of z that were recorded from the FEMLAB simulation.

   An array function, Lev (Table 1) was written to evaluate Eq (2) and Eq (3). The values of the Gamma function at 4/3, 2/3, and 5/3 were taken from [8]. For larger values of the Gamma function [9] the relationship

   $$\Gamma\ (x) = (x-1) * \Gamma\ (x-1) \qquad (4)$$

   was used. The output of the array function, Lev, is indicated in Fig (3) by the boxed highlight.

In order to test the array function two methods of specifying z were used. Either the z value could be directly specified or it could be evaluated from the value specied for Z* x 10^5 and the results compared to the published values of 1/(1-Td) indicated in [8] for various values of λ.

Results of the Study

The data in the table of Fig (3) is plotted in Figs (4) and (5). The FEMLAB temperature at the wall lies below the theoretical (Leveque) results but the conductive flux lies above the theoretical results.

It is not known if a finer mesh would result in a value of the FEMLAB conductive flux (at z =0, r= a) closer to the theoretical value [11, 007 J/(s-m^2)] equal to k λ $(t_d - t_o)$ /a [see Eq (3)]. Note that at r=a, the value of of $k_s * (t_d\text{-}t)/(a \ln(d/a))$ where $t = t_o$ is 10,955 J/(s-m^2). This boundary specification in FEMLAB is in contrast to indicated value of 20000. The disparity is thought to be due to the temperature discontinuity at the entrance wall.

Discussion of Results

The comparison of the theoretical solution to the FEMLAB solution is quite good except at the entrance. The assumptions of no buoyancy effects and flat plate geometry seem reasonable but were not confirmed and remain assumptions.

Conclusions

Given a small enough mesh size, FEMLAB can reasonably reproduce the analytical solution in regions of rapid change. For such solutions, however, adequate memory is required.

Nomenclature

English

a	Inside tube radius- m
c_p	Heat capacity- J/(kg-K)
d	Outside radius of tube- m
G	Mass flow rate- kg/(s-m^2)
k	Thermal Conductivity of fluid- J/(s-m-K)
k_s	Thermal conductivity of glass tubing- J/(m-s-K)
n	An index
Pr	Prandtl Number - $(c_p\ \mu)/k$
$q_{w/}/A$	Conductive flux at the wall- J/(s-m^2)
r	Radial distance- m
Re	Reynolds number- $(a\ G)/\mu$
t	Temperature- K
t_w	Temperature of the wall- K
t_o	Temperature of entering fluid- K
t_d	Temperature of outside tube wall- K
T_d	$(t_d – t_w)/t_o – t_w)$
z	Distance downstream- m
Z	z/a
Z^*	$Z/(\pi\ Re\ Pr\ )$

Greek

η	Parameter equal to $\frac{\lambda\, 18^{1/3}\, \Gamma(4/3)\, \pi^{1/3}}{2\, \Gamma(2/3)}$
Γ	Gamma Function
λ	Parameter equal to $k_s/[k \ln(d/a)]$
μ	Viscosity- kg/(m-s)

References

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3. Rosen, E. M., “Effect of Heat Transfer on Flow Field At Low Reynolds Numbers in Vertical Tubes” Ph. D. Thesis in Chemical Engineering, University of Illinois, 1959

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6. The MathWorks, Inc. 3 Apple Hill Drive, Natick, MA 01760-2098, http://www.mathworks.com

7. COMSOL,Inc. 8 New England, Executive Park, Burlington, MA 01803 http://www.comsol.com

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9. http://www.efunda.com/

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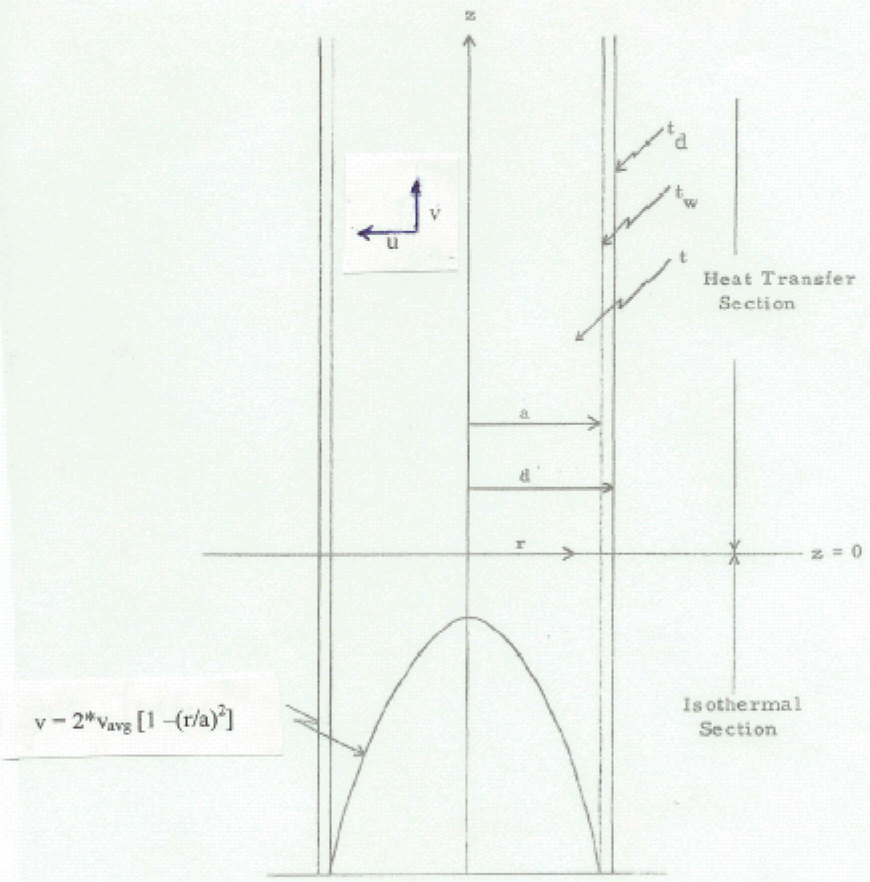

Figure 1. Definition of Coordinate System and Nomenclature

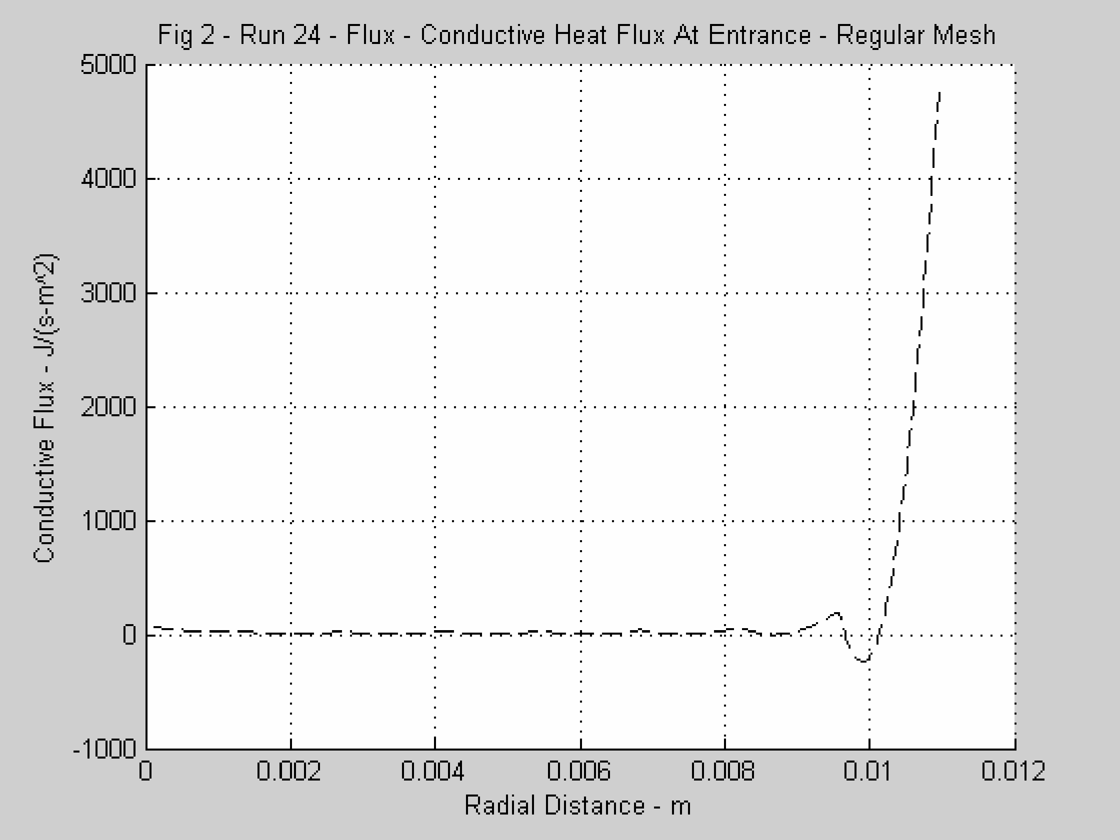Fig 2 - Run 24 - Flux - Conductive Heat Flux At Entrance - Regular Mesh
Conductive Flux - J/(s-m^2)
5000
4000
3000
2000
1000
0
-1000
0
0.002
0.004
0.006
0.008
0.01
0.012
Radial Distance - m

Figure 3 Spreadsheet for Evaluation of Wall Temperature and Conductive Flux at Wall for Run 24

				__Femlab Run 24__	Lambda = 13.8	__Femlab - Run 24__	Lambda = 13.8
			__z__	Wall temp - Femlab	Wall temp - Leveque	Wall Flux - Femlab	Wall Flux - Leveque
			__meters__	Kelvins	Kelvins	J/(s-m^2)	J/(s-m^2)
___Set = 1- B7, Set 0 B46___	1	*					
z (specified) z	0.000025	*	0	300.983	300.983	20000	11007.986
		*	2.5E-05	301.7	303.17	18000	9300.11
z (m)	0.000025	*	0.00005	302.4	303.64	15500	8932.15
Gamma-4/3	0.892979	*	0.0001	303.1	304.19	11500	8504.56
Gamma-2/3	1.35411	*	0.0002	304.1	304.82	9700	8016.27
Re	164.213	*	0.0003	304.5	305.22	8800	7703.36
Pr	5.6645	*	0.0004	304.9	305.51	8100	7469.86
Pi	3.14159265	*	0.0005	305.2	305.75	7600	7282.62
Z	0.002283105	*	0.0006	305.4	305.95	7500	7125.9
__Z* x 10^5__	0.078128035	*	0.0007	305.6	306.13	7450	6990.97
		*	0.0008	305.8	306.28	7300	6872.38
___Parameters___		*	0.0009	305.95	306.42	7150	6766.56
		*	0.001	306.1	306.54	7050	6766.56
Z*	7.8128E-07	*	0.0011	306.2	306.65	6940	6583.85
a	0.01095	*	0.0012	306.3	306.75	6850	6503.75
Value of n	150	*	0.0013	306.45	306.85	6760	6429.65
Lambda	13.8	*	0.0014	306.52	306.94	6670	6360.7
td	315.094	*	0.0015	306.6	307.02	6610	6296.23
to	300.983	*	0.0016	306.7	307.1	6550	6235.7
k	0.618989	*	0.0017	306.72	307.17	6500	6178.66
eta	17.4653221	*	0.0018	306.74	307.24	6490	6124.72
		*	0.0019	306.71	307.31	6514	6073.58
		*	0.002	306.68	307.37	6660	6024.95
___Return from Lev___		*					
		*					
__1/(1-Td)__	0.155144248	*					
__flux__	9300.113643	*					
		*					
___Calculate from Return___		*					
		*					
__tw__	303.1722405	*					
		*					
		*					
Choose Z*		*					
Calculate z from Z*		*					
		*					
__Z* x 10^5__	4	*					
Z*	0.00004	*					
Z	0.116890436	*					
z	0.00127995	*					

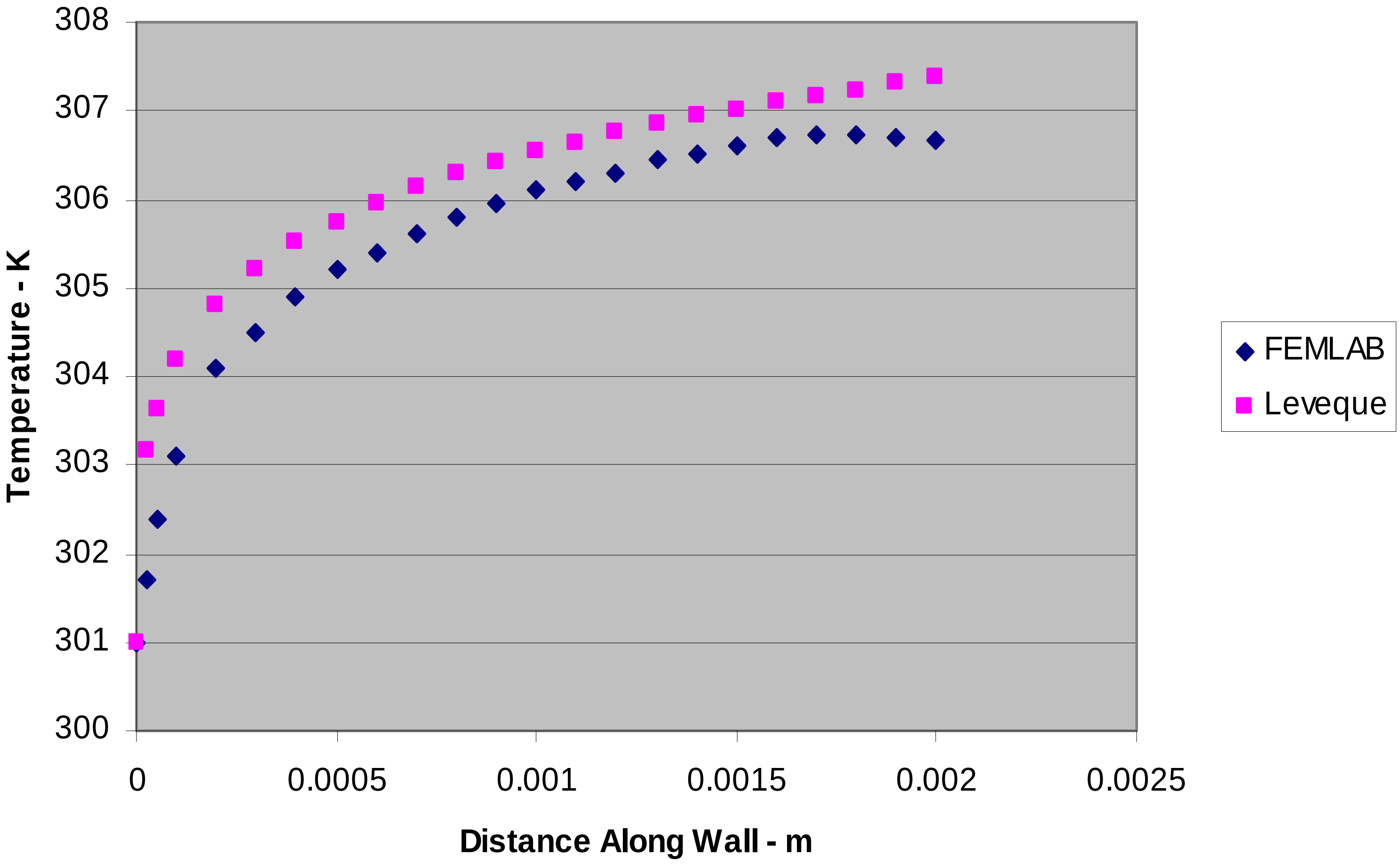

Fig 4 -Run 24 - Flux -Temperature At Wall Near Entrance

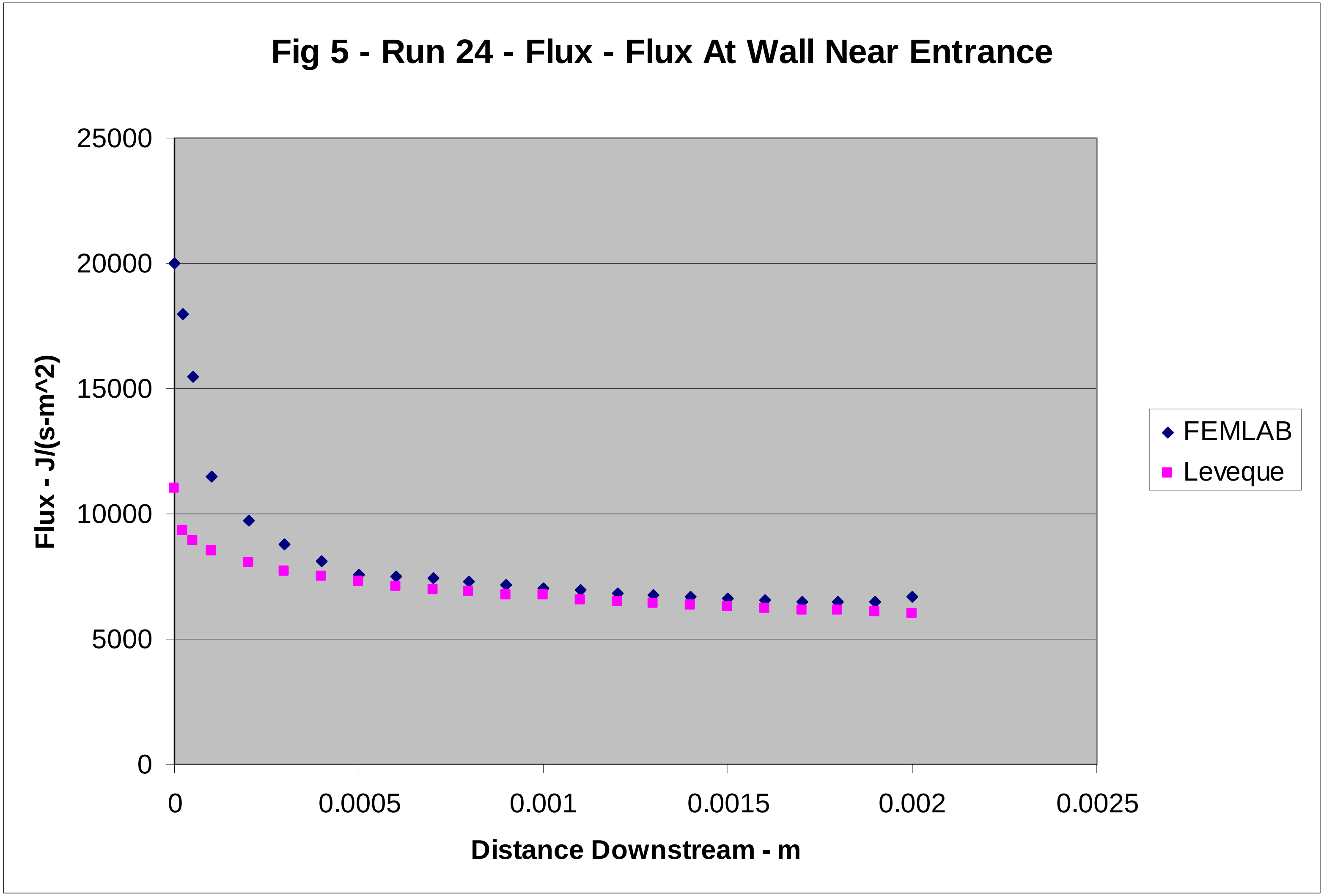

Fig 5 - Run 24 - Flux - Flux At Wall Near Entrance

```
Option Explicit

Public Function Lev(p)

Application.Volatile True

Dim n, I As Integer

Dim xn, a, lambda As Double
Dim zstar, eta, k As Double

Dim Arg, Gamf As Double
Dim Sum, Sumf, t0, td As Double

Dim Value(1 To 200) As Double
Dim Out(1 To 2) As Double

Const gam4t = 0.89297965338, gamtt = 1.35411, game = 1#, gam5t = 0.902745913454

zstar = p(1)
a = p(2)
n = p(3)
lambda = p(4)
td = p(5)
t0 = p(6)
k = p(7)
eta = p(8)

Sum = 0#
Sumf = 0#

For I = 1 To n

   xn = I
   Arg = 1 + xn / 3

   If I = 1 Then
      Value(1) = gam4t
      Gamf = Value(1)
   End If


  If I = 2 Then
     Value(2) = gam5t
     Gamf = Value(2)
  End If


  If I = 3 Then
      Value(3) = game
      Gamf = Value(3)
  End If

   If I > 3 Then
       Value(I) = (Arg - 1) * Value(I - 3)
           Gamf = Value(I)
   End If
```

```
 Sum = Sum + (-1) ^ (I + 1) * ((eta * zstar ^ 0.33333) ^ I) / Gamf

Sumf = Sumf + (-1) ^ (I) * ((eta * zstar ^ 0.33333) ^ I) / Gamf


Next I

Out(1) = Sum
Out(2) = k * lambda * (td - t0) * (1 + Sumf) / a

Lev = Application.Transpose(Out)

End Function
```

Table 1
Listing of Array Function Lev