Femlab 3.1: The Vertical Heated Plate

by

Edward M. Rosen

EMR Technology Group

A vertical heated plate exposed to air has been the subject of experiments, analysis and academic exercises (Goldstein, 2004, Yang, 2002). It is the purpose of this study to compare Femlab 3.1 simulation results to the analytical solution and data.

Problem Description

An isothermal vertical plate at temperature T_w is exposed to air at T_∞. The vertical distance along the plate is the x direction and its velocity is u. The horizontal direction perpendicular to the plate is the y direction and its velocity is v.

For this study T_w = 115 C and T_∞ = 20 C . Physical properties are evaluated at the average temperature, 67 C and are considered constant. These temperatures are chosen to be consistent with selected data (McAdams, 1955).

The Analytical Solution

After simplifications the applicable (boundary-layer) equations are (Bejan, 1995, Boelter, 1948, Goldstein, 2004, Kays, 1993, Schlichting, 1955)

Continuity
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Hydrodynamic
$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \upsilon \frac{\partial^2 u}{\partial y^2} + g \frac{T_w - T_\infty}{T_\infty} \theta \qquad (2)$$

Energy
$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \qquad (3)$$

Introducing a stream function ψ by putting u = $\frac{\partial \psi}{\partial y}$ and v = $-\frac{\partial \psi}{\partial x}$ and substituting into Eqs (1 -3) results in

$$\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \tag{4}$$

$$\frac{\partial \psi}{\partial y}\,\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\,\frac{\partial^2 \psi}{\partial y^2} = \upsilon\,\frac{\partial^3 \psi}{\partial y^3} + g\,\frac{T - T_\infty}{T_\infty} \tag{5}$$

$$\frac{\partial \psi}{\partial y}\,\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\,\frac{\partial \theta}{\partial y} = \alpha\,\frac{\partial^2 \theta}{\partial y^2} \tag{6}$$

Setting (Schlichting, 1955)

$$\psi = 4\,\upsilon\, c\, x^{3/4}\;\xi\,(\eta) \tag{7}$$

$$\eta = c\,\frac{y}{\sqrt[4]{x}} \tag{8}$$

where

$$c = \sqrt[4]{\frac{g\,(T_w - T_\infty)}{4\,\upsilon^2\,T_\infty}} \tag{9}$$

the velocity components become

$$u = 4\,\upsilon\, x^{1/2}\; c^2\;\xi' \tag{10}$$

$$v = \upsilon\, c\, x^{-1/4}\;(\eta\,\xi' - 3\,\xi)$$

and the temperature distribution is determined by $\theta\,(\eta)$. Substituting Eq (7) into Eqs (5) and (6) leads to the following ordinary differential equations:

$$\xi''' + 3\,\xi\,\xi'' - 2\,\xi'^2 + \theta \;= 0 \tag{11}$$

$$\theta'' + 3\,\Pr\,\xi\,\theta' = 0 \tag{12}$$

with boundary conditions $\xi = \xi' = 0$ *and* $\theta = 1$ *at* $\eta = 0$ *and* $\xi' = 0$, $\theta = 0$ *at* $\eta = \infty$.

Eqs (11) and (12) can be formulated as a set of five first order differential equations (Goldstein, 2004)

$$\frac{d\xi}{d\eta} = p_1 \qquad\qquad \xi = 0 \; at \; \eta = 0$$

$$\frac{dp_1}{d\eta} = p_2 \qquad\qquad p_1 = 0 \; at \; \eta = 0$$

$$\frac{dp_2}{d\eta} = -3\,\xi\,p_2 + 2\,p_1^{\,2} - \theta \qquad\qquad p_2 \; unknown \text{ at } \eta = 0$$

$$\frac{d\theta}{d\eta} = q \qquad\qquad \theta = 1 \; at \; \eta = 0$$

$$\frac{dq}{d\eta} = -3\,\xi\,\Pr\,q \qquad\qquad \text{q unknown at } \eta = 0 \tag{13}$$

Since p_2 and q are unknown at η =0 but known to be = 0 at η = ∞ (taken as η = 12), this is a two point boundary value problem (Goldstein, 2004). The values of p_2 and q at η = 0 have been tabulated as a function of Pr (Kakac et. al., 1985) and are shown in Table 1.

Physical Properties

Values for the physical properties of air at 67 C were taken from the *Fluid Properties Calculator* which can be found at

http://www.mhtl.uwaterloo.ca/old/onlinetools/airprop/airprop.html

Prepared by Ramesh Krishnamurthy
Kakac et al 1985

Pr	$p_2(0)$	q(0)	$\xi(\infty)$
0.01	0.9873	-0.0807	4.536
0.1	0.8591	-0.2301	1.512
0.72	0.676	-0.5046	0.5988
1	0.6422	-0.5671	0.5253
2	0.5713	-0.7165	0.4046
5	0.4818	-0.954	0.3031
6.7	0.4548	-1.0408	0.2786
10	0.4192	-1.1693	0.2792
100	0.2517	-2.1913	0.1365

Table 1 Values of p_2 and q at $\eta = 0$

Spreadsheet Calculations

The physical properties of air are recorded on a spreadsheet (Appendix A) and Pr is calculated. The values of p_2 and q are then interpolated from the values given in Table 1 after converting Pr to ln Pr. This allows the integration of Eqs (13) as shown in the spreadsheet. The integration was carried out in increments of $\eta = 0.025$ up to $\eta = 12$. (The spreadsheet shows values of η only up to 0.575)

The values of u and v at each value of η were calculated from Eqs (10) for x = 0.06 m.

The Femlab Solution

A tutorial " heated plate" is supplied by Femlab, 2004 (Version 3.1) and generally guided the Femlab solution in this study.

Some modifications were made, however :

1. Femlab uses the the Boussinesq approximation $\rho = \rho_\infty \left(1 - \frac{T - T_\infty}{T_\infty} \right)$

   in the tutorial. This value of ρ is used in both the energy and hydrodynamic equations. The buoyancy term is $F_x = \rho_\infty \left(\frac{T - T_\infty}{T_\infty} \right) g$.

   Choosing T_∞ as the temperature at which to evaluate the physical properties (i.e. ρ_∞) in the analytical solution results in the same buoyancy term as in Femlab. However ρ from the Boussinesq equation is also used in the energy equation in Femlab while the value of ρ in the analytical solution energy equation would be ρ_∞.

   As a result the Boussinesq equation was not used in this study's Femlab simulation. A constant ρ was used throughout instead at 67 C. This makes the density used in Femlab the same as used in the analytical solution.

2. The geometry was changed to have the plate on the left side rather than the right.

3 In order to compare the Femlab solution to the analytical, the conditions far from the plate should be the same. The boundary conditions in the tutorial are indicated in Table 2. However, they do not adequately satisfy the conditions far from the plate:

   a) The velocity (u) should drop to zero far from the plate.

   b) The value of the velocity, v, (perpendicular to the plate) should be equal to -0.0153 far from the plate. This derives from evaluation of v from Eq (10) at $\eta = \infty$.

      $v = c\, x^{-1/4} (-3\xi) = -0.0153$ since $\xi' = 0$ and $\xi(\infty) = 0.603201$ (Table 1) at $\eta = \infty$. ($x = 0.06$)

By trial, the boundary values used in this study to meet the conditions far from the plate are shown in Table 2.

Description	No.	***Femlab Tutorial***		***This Work***	
		Thermal	**Flow**	**Thermal**	**Flow**
Lower Insulated	1	Thermal Insulation	No slip	Thermal Insulation	No slip
Bottom	2	Tinf	***Neutral***	Tinf	***No slip***
Plate	3	Tinf + dT	No slip	Tinf + dT	No slip
Upper Insulated	4	Thermal Insulation	No slip	Thermal Insulation	No slip
Top	5	***Convective Flux***	***Neutral***	***Tinf***	***No slip***
Far Boundary	6	Tinf	Neutral	Tinf	Neutral

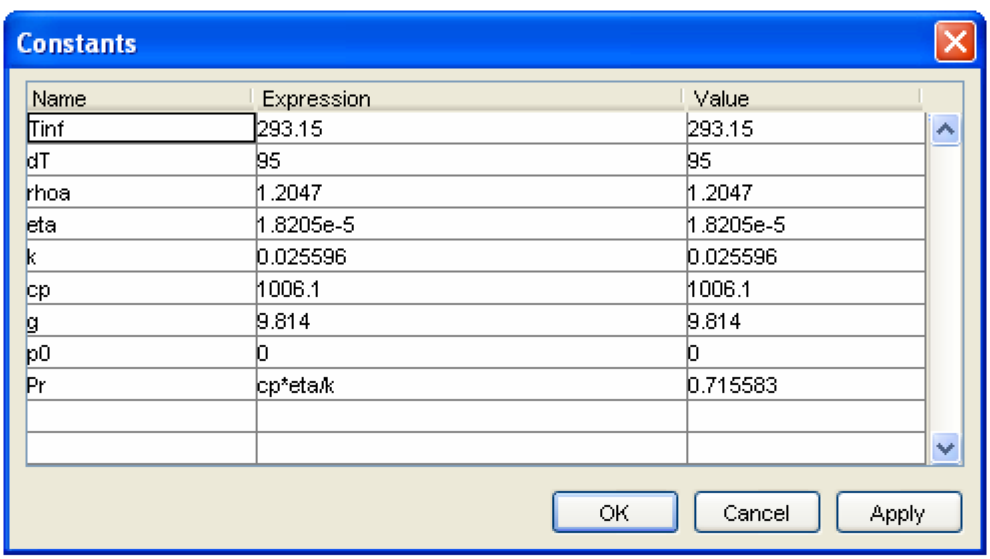
Table 2 - Boundary Conditions Compared

Model Navigator

1. Start Femlab – Set Space Dimension to 2D
2. Select *General Heat Transfer* application in the *Heat Transfer Module*
3. Click *Steady State Analysis.*
4. Click the *Multiphysics* button in the *Model Navigator*.
5. Click the *Add* button.
6. Select the *Non-Isothermal Flow* in the *Heat Transfer Module*
7. Click the *Add* button
8. Click *OK*

Options and Settings

1. Go to the *Options* menu and define the following physical properties at 67 C.



Name	Expression	Value
Tinf	293.15	293.15
dT	95	95
rhoa	1.2047	1.2047
eta	1.8205e-5	1.8205e-5
k	0.025596	0.025596
cp	1006.1	1006.1
g	9.814	9.814
p0	0	0
Pr	cp*eta/k	0.715583

Figure 1 Constants

2. Click *OK*.

Geometry Modeling

Fig (1) is the two dimensional geometry of the heated plate. The plate is on the left between PT1 and PT2 , is 0.1 m tall and has isothermal temperature T_w. The short boundaries (0.01 m) below and above the vertical plate are thermally insulated and are needed to create a smooth flow near the plates edges. On the bottom, top and right boundary the temperature is set to T_∞.

Air will rise upwards towards the upper horizontal boundary. At the top boundary the no slip specification forces the flow out to the far boundary. The neutral boundary condition means that no forces act on the fluid.

1. Press the *Shift* key and click *Rectangle/Square* button in the *Draw* toolbar.
2. In the dialog box that appears, enter the following rectangle properties:

   Width 0.105
   Height 0.12
   Base Corner
   x-position 0 y position 0

3. Click *OK*
4. Click *Zoom Extents* in the Main Toolbar
5. Press the *Shift* key and click the *Point* button in the Draw toolbar.
6. In the dialog box that appears, enter the following point properties (PT1):

   x 0
   y 0.01

7. Click *OK*

8. In the same manner, create a second point (PT2):

   x 0
   y 0.11

9. Click *OK*

Note: In Femlab x is the horizontal direction, y is the vertical direction,
Also note that 0.07 m from the bottom of the geometry is 0.06 m from the bottom of the plate.

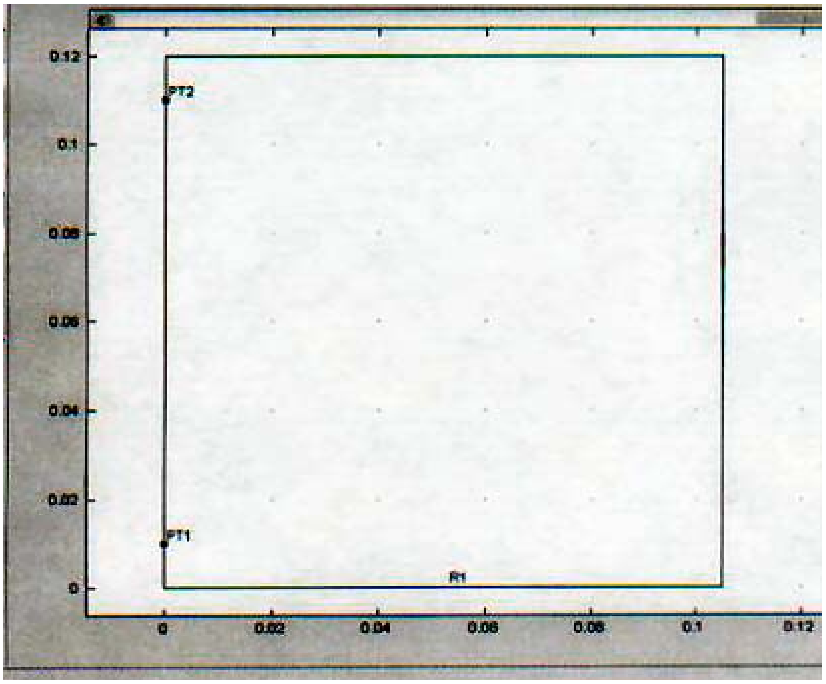


Figure 2. Heated Plate Geometry
T_w = 115 C

Physics Settings

1. Go to the *Options* menu, find *Expressions* and then open *Subdomain Expressions*

   Enter Expression for FY

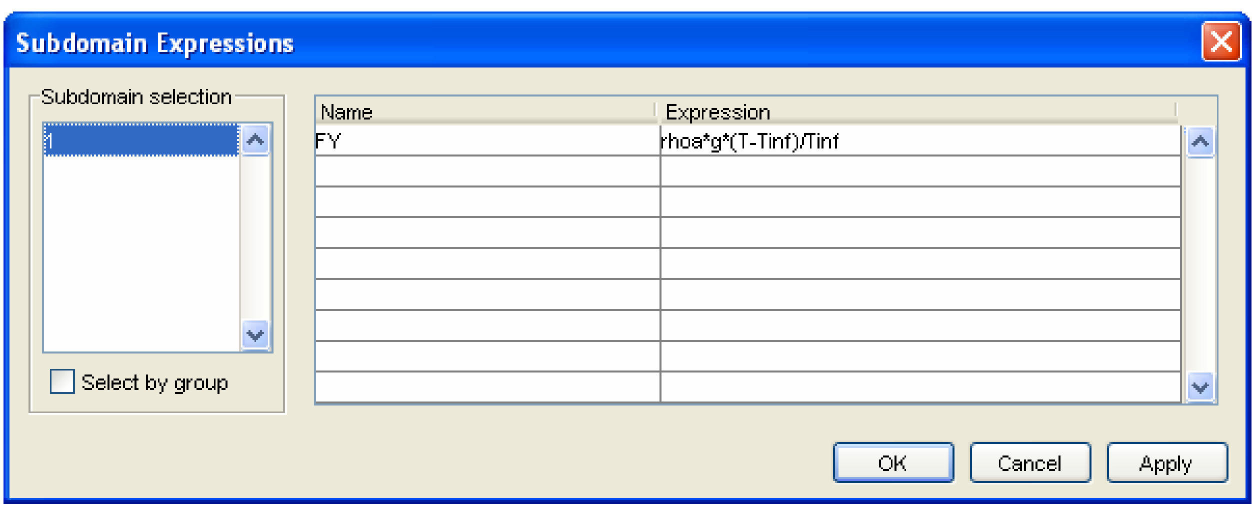


Figure 3 Subdomain Expression

Subdomain Settings – Non-Isothermal Flow

1. Go to the *Multiphysics* menu and select *Non-Isothermal Flow*
2. Go to the *Physics* menu and select *Subdomain Settings*
3. Enter the following

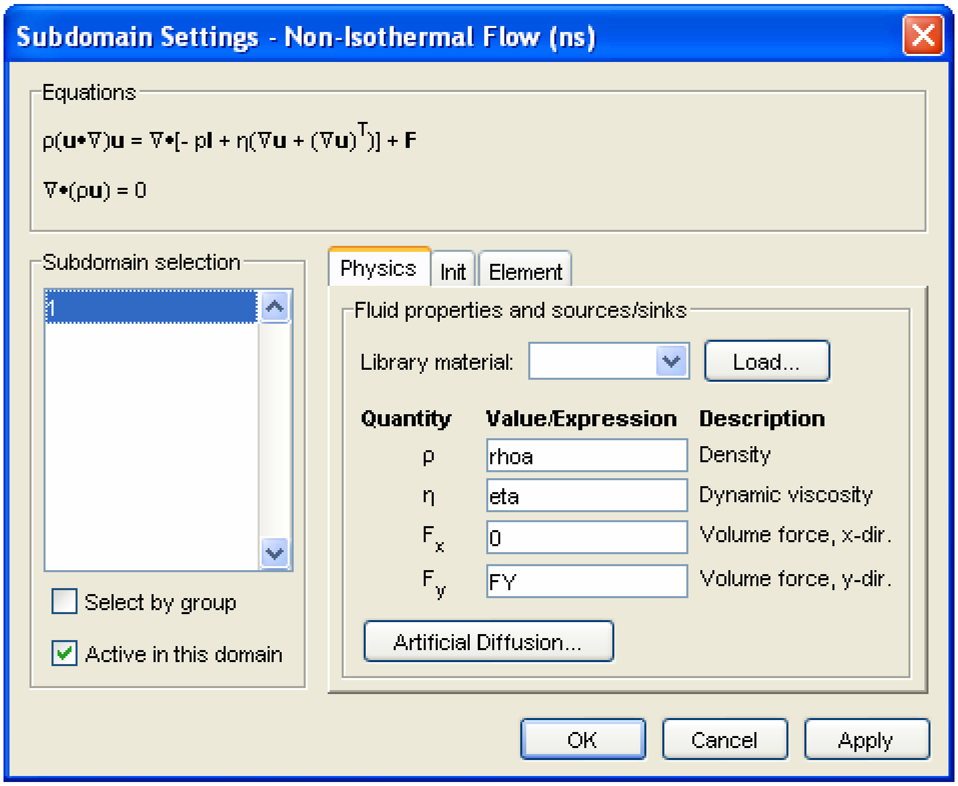


Figure 4 Subdomain Settings – Non-Isothermal Flow (ns)

4. Click *OK*

Boundary Settings – Non-Isothermal Flow

1. Go to the *Physics* menu and select *Boundary Settings*
2. Enter the following*:*

Boundary 1	Short lower left side	-	*No Slip*
Boundary 2	Bottom	-	*No Slip*
Boundary 3	Heated Plate	-	*No Slip*
Boundary 4	Short top left side	-	*No Slip*
Boundary 5	Top	-	*No Slip*
Boundary 6	Far right side	-	*Neutral*

3. Click *OK*

Subdomain Settings – General Heat Transfer

1. Go to the *Multiphysics* menu and select *General Heat Transfer*
2. In the Physics menu, select *Subdomain Settings*
3. Select subdomain 1 and verify the *Enable conductive heat transfer* is checked.
4. Enter the properties for *Conduction* as shown:

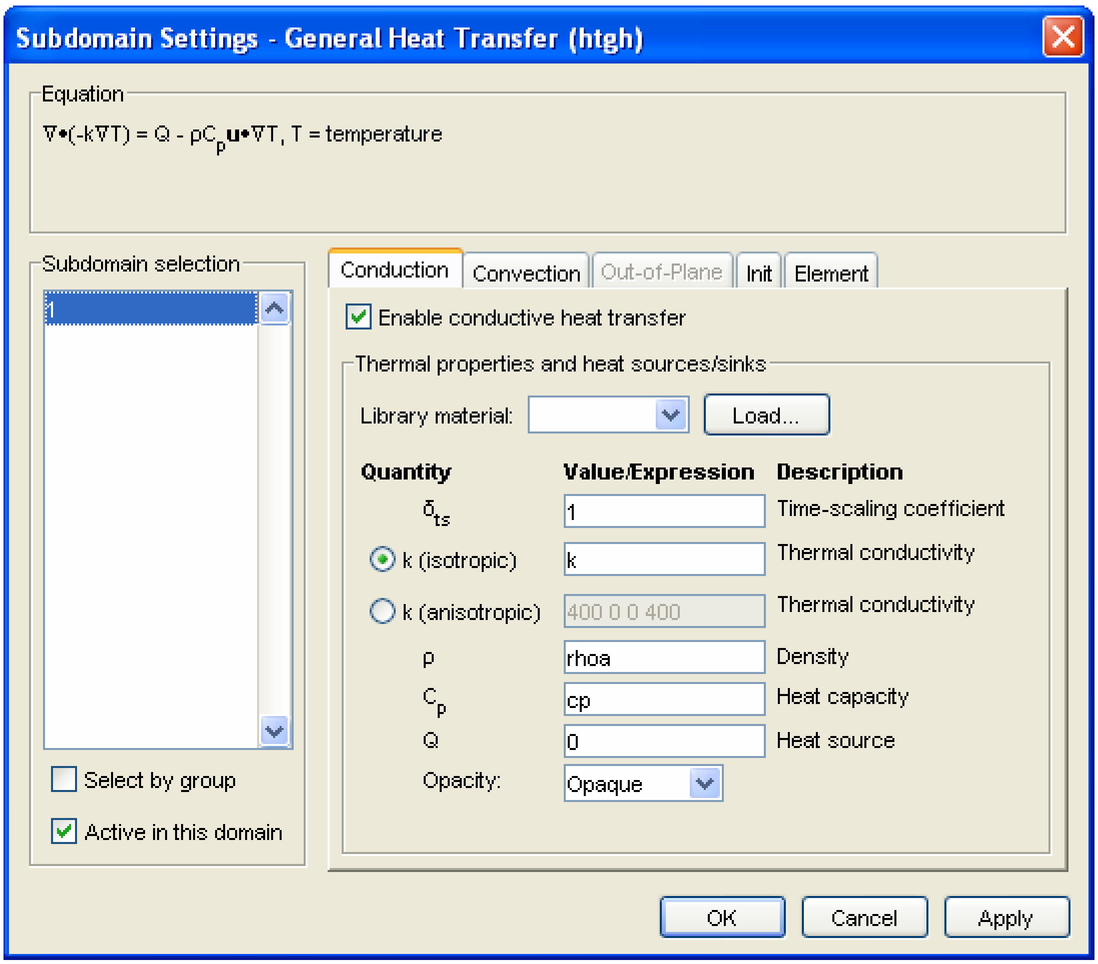


Figure 5. Subdomain Settings – General Heat Transfer (htgh) - Conduction

5. Click the *Convection* tab and select *Enable convective heat transfer*.
6. Enter the properties as shown.

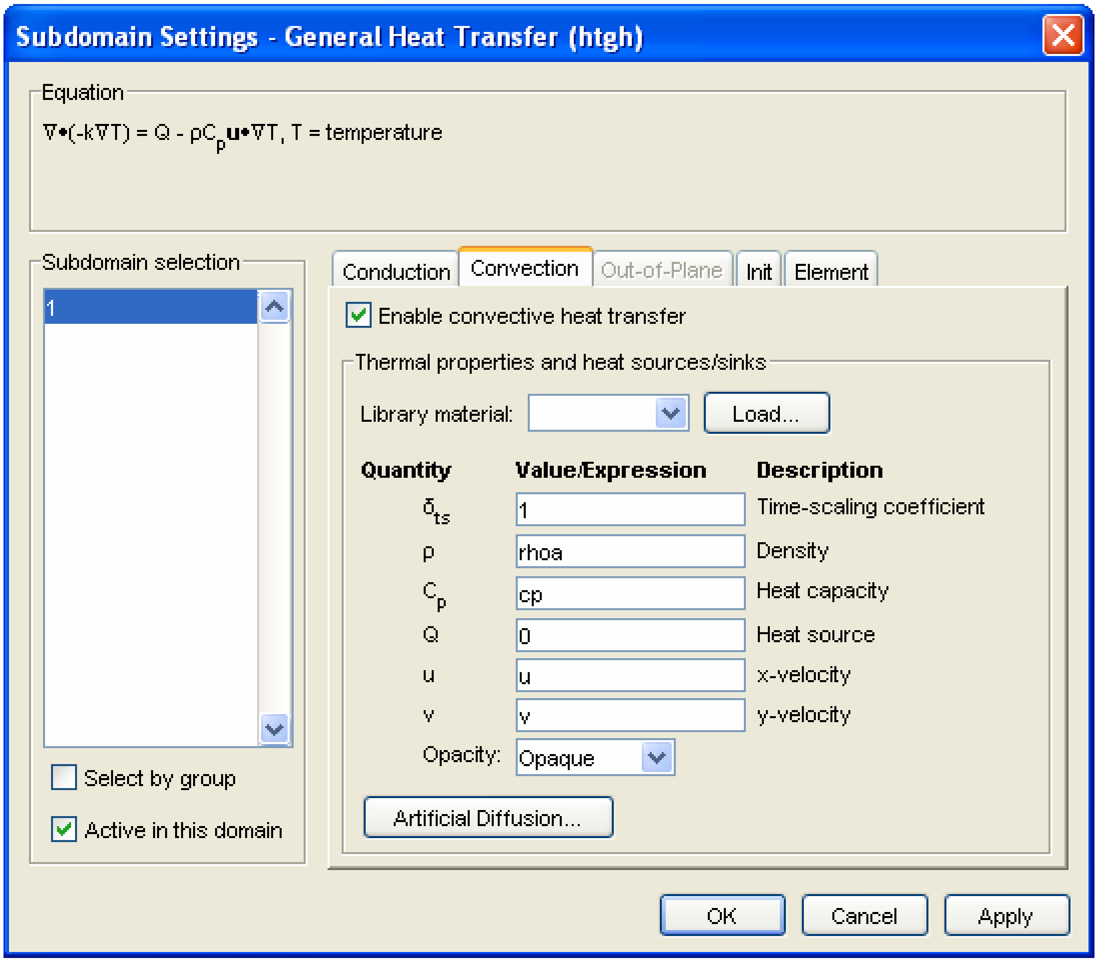


Figure 6 Subdomain Settings – General Heat Transfer (htgh) - Convection

7. Click the *Init* tab.
8. Sect *subdomain* 1 and type Tinf in the temperature edit field.
9. Click *OK*

Boundary Conditions – General Heat Transfer

1. Go to the *Physics* menu and enter the following *Boundary Settings*
2. Enter the following.

Boundary 1	Short lower left side	-	Thermal Insulation
Boundary 2	Bottom	-	Temperature - Tinf
Boundary 3	Heated Plate	-	Temperature - Tinf + dT
Boundary 4	Short top left side	-	Thermal Insulation
Boundary 5	Top	-	Tempersture Tinf
Boundary 6	Far right side	-	Temperature Tinf

3. Click *OK*

Mesh Generation

1. From the *Mesh* menu select *Mesh Parameters*.
2. Click the *Boundary* tab and select boundaries 1, 3, 4
3. Enter 3e-4 in the *Maximum element size* edit field.
4. Click the *Point* tab and select point 2 (PT1)
5. Enter 2e-5 in the *Maximum element size* edit field.
6. Click *Remesh* and then click *OK*.

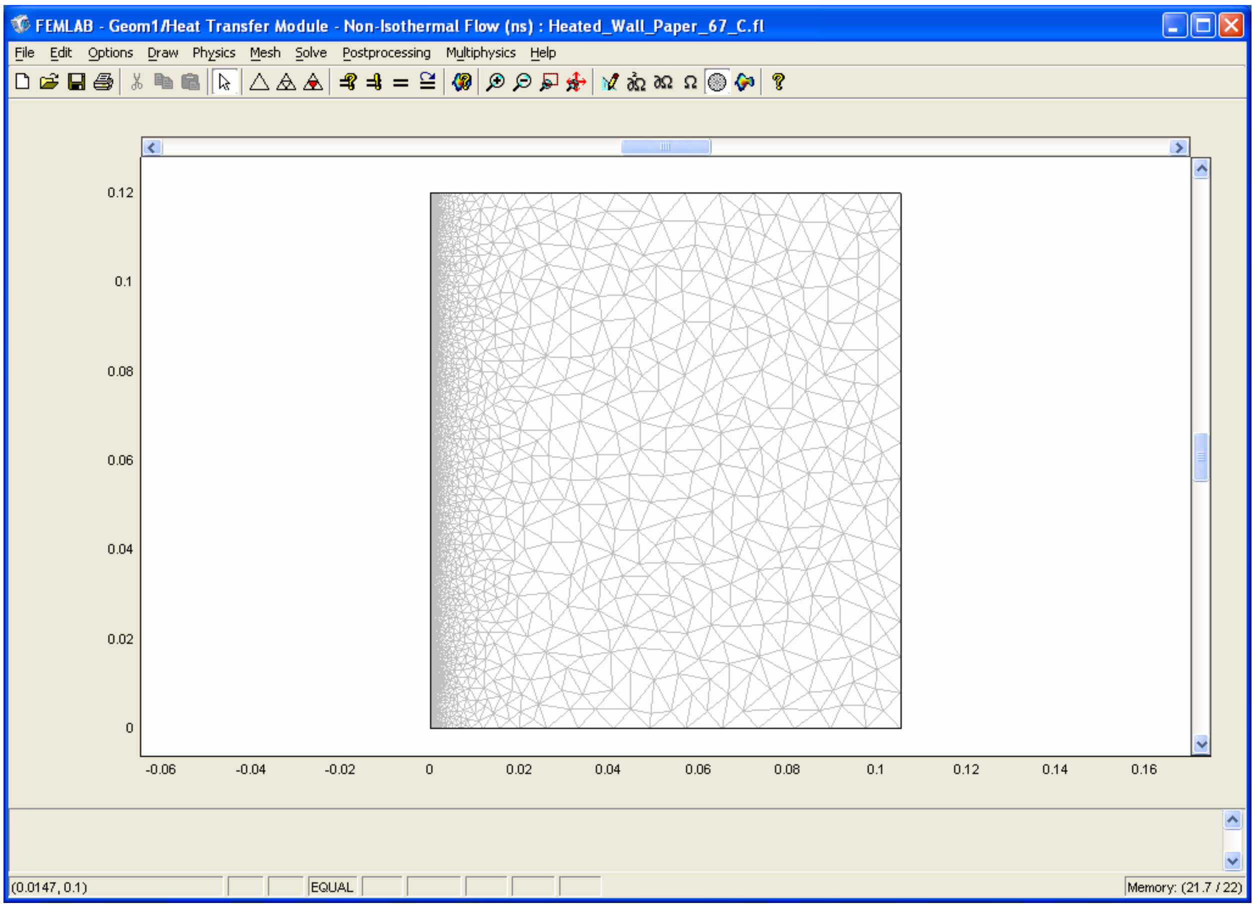


Figure 7. Mesh Generation

Computing the Solution

Finding a converged solution is difficult. It is best approached using a small dT (say 10 K) at first and slowly decreasing the value of viscosity from about 10 in steps down to the specified viscosity using the Parametric nonlinear Solver. Once the solution with the required viscosity is found, then the value of dT is increased to the desired value (95) again using the Parametric nonlinear Solver.

1. Reset the value of dT in the *Constants* window to 10.
2. Choose *Solver Manager* from the *Solve* menu.
3. On the *Initial* value page, select *Initial Value Expression* and *Use setting from Initial Value Frame*
4. Choose *Solver Parameters* from the *Solve* menu.
5. In the dialog box that appears, select *Parametric Nonlinear* in the *Solver* list.
6. On the *General* page, enter eta (dynamic viscosity) into the *Name of the parameter* edit field
7. Type 10 1 .1 1e-2 1e-3 1e-4 1.8205e-5 into *List of parameter values* edit field.
8. On the *Advanced* page, set *Type of scaling* to *None*.
9. Click *OK*
10. On the *Solve* menu, click *Solve Problem*

The desired solution (dT = 95) is now sought with the eta = 1.8205e-5.

1. Open the *Solver Manager* from the *Solve* menu.
2. On the *Initial Value* page, click *Store solution.*
3. Select the value 1.8205e-5 and click *OK.*
4. On the *Initial value* page, select *Stored solution* for both *Initial Value Expression* and *Use setting from Initial Value Frame.*
5. Click *OK.*
6. Choose *Solver Parameters* from the *Solve* menu.
7. On the *General* page, enter dT in the *Name of parameter* edit field
8. Type 10 50 90 95 into the *List of parameter values* edit field.
9. Click *OK.*
10. On the *Solve* menu, click *Solve Problem.*

When the solution is obtained the following results:

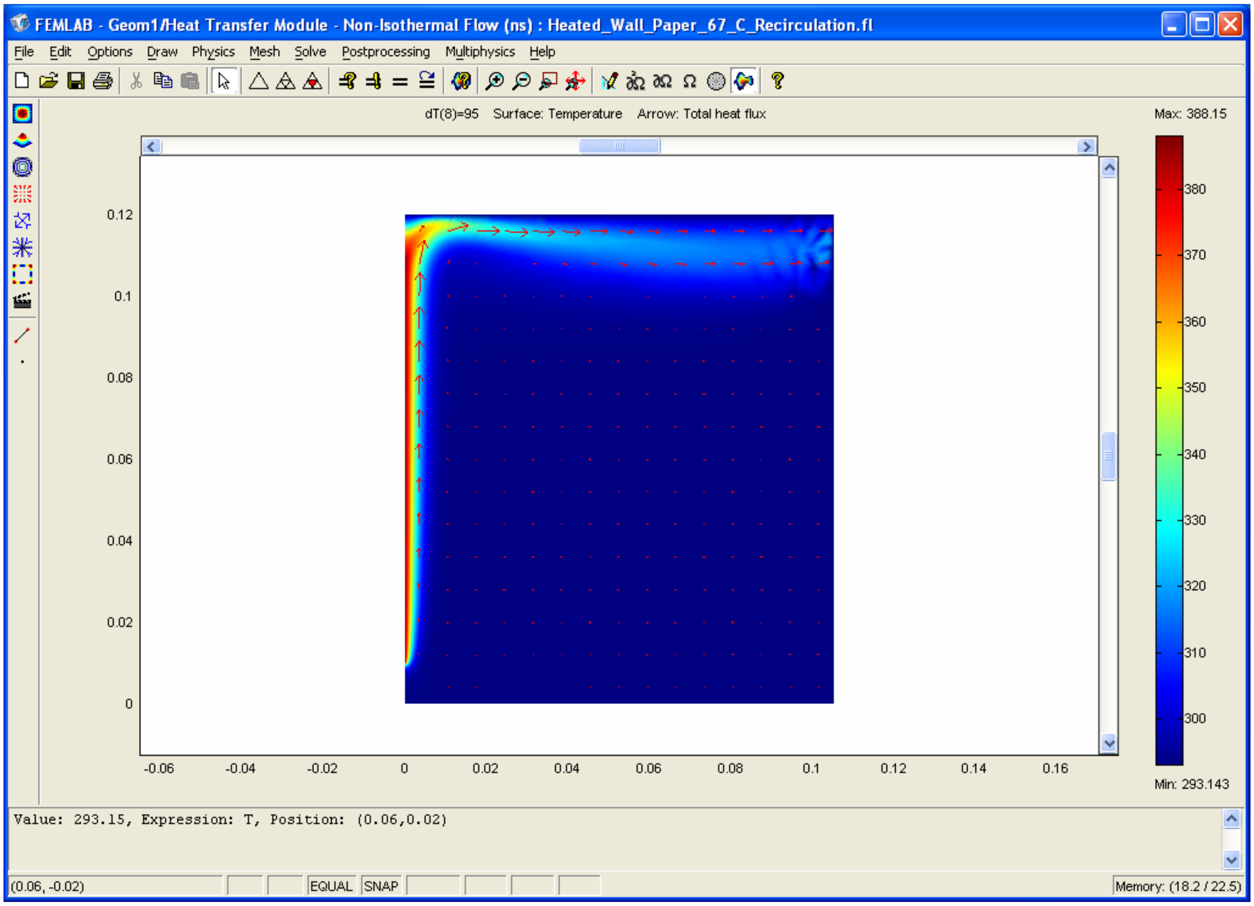


Figure 8. Surface Temperature and Arrows Showing Flows

Comparing Femlab, The Analytical Solution and Data

Femlab

The velocity, temperature and temperature gradient profiles are displayed at 0.06 m from the plate bottom (0.07 m in Femlab) using the Femlab postprocessing procedures. In each case the ASCII output capability is utilized to obtain a numerical listing in an ASCII file. The ASCII files are then imported into a spreadsheet. For plotting, interpolation [add-in function Interp (Rosen, 2004)] is used to obtain the values of interest at pre-determined values of y or x.

Analytical Solution

Values of the velocity and temperature profiles at 0.06 m from the bottom of the plate are taken from the spreadsheet solution (Appendix A). The temperature gradient along the plate at a specified value of x is calculated on the spreadsheet from:

$$\left[\frac{\partial T}{\partial y}\right]_{y=0} = \left[\frac{Gr_x}{4}\right]^{1/4} \left[\frac{\partial \theta}{\partial \eta}\right]_{\eta=0} \frac{(T_w - T_\infty)}{x} \qquad (14)$$

Data

Values for the velocity profile at at 0.06 m are read directly from Fig 123 (McAdams, 1955). Values for the temperature profile at 0.06 m are interpolated from the values at 0.01 m, 0.04 m, 0.09 m, and 0.14 m.

Values from the spreadsheet for the analytical solution (Appendix A), the Femlab solution and data points are shown in Figs (9-12).

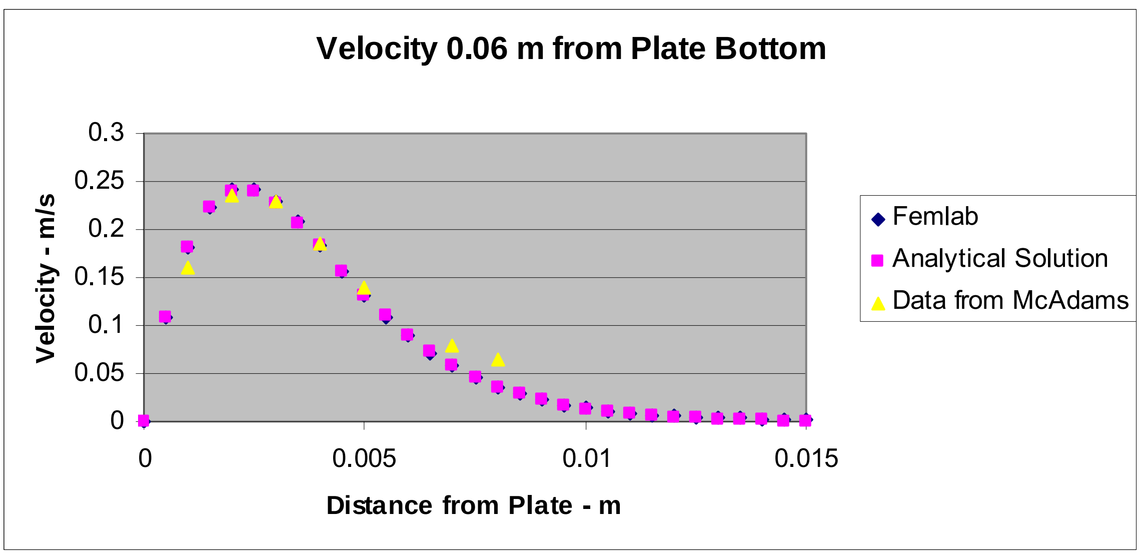


Fig 9. Velocity Profile at 0.06 m
$T_w - T_\infty = 95\ C$

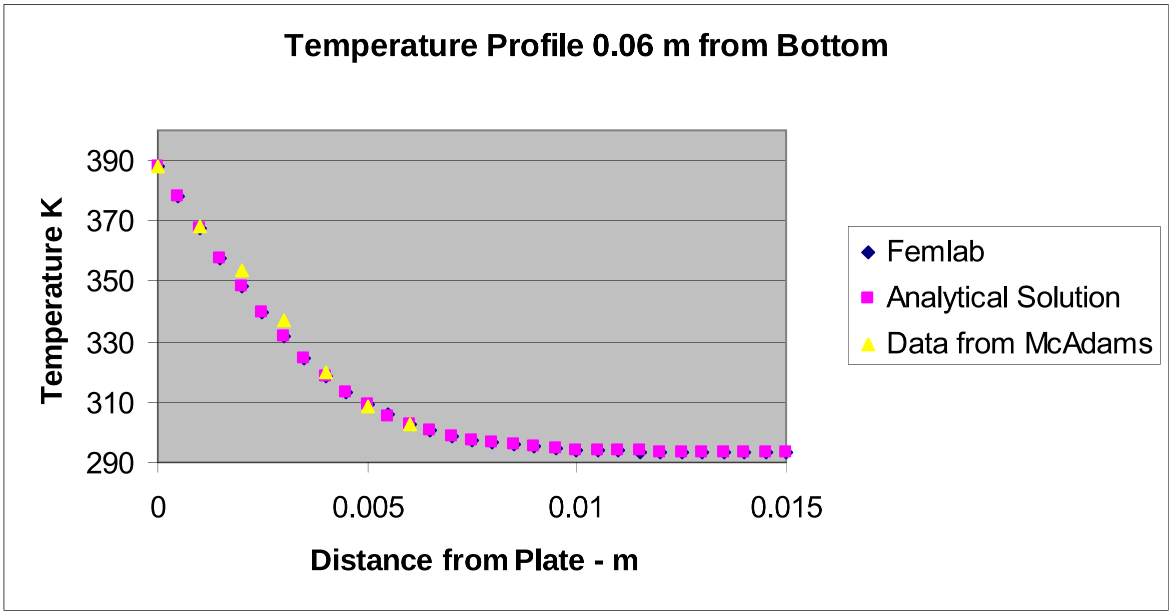


Figure 10. Temperature Profile at 0.06 m
$T_w - T_\infty = 95\ C$

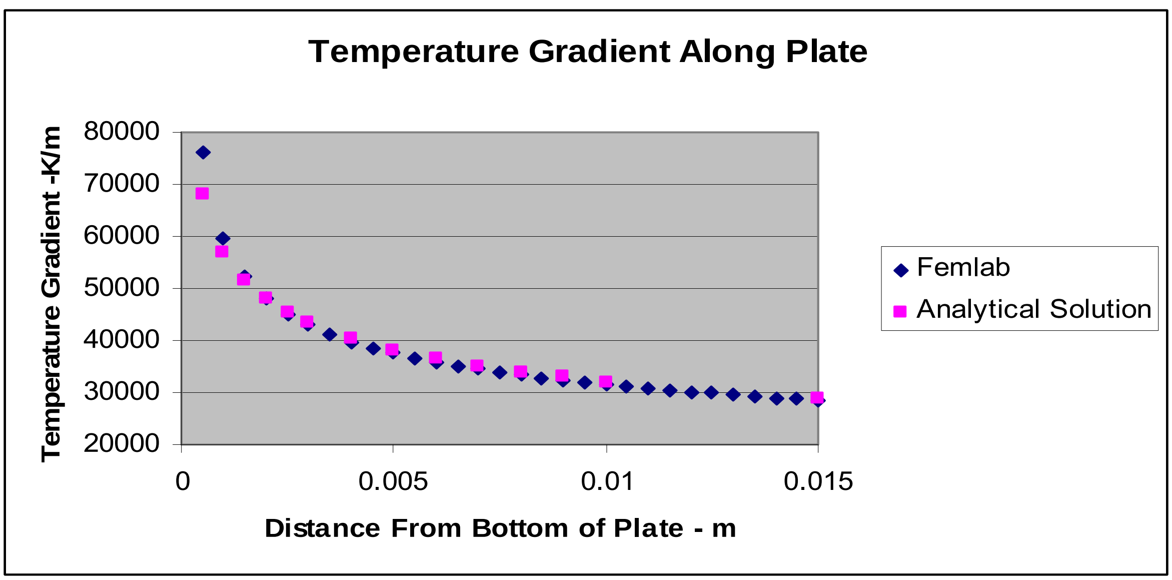


Figure 11. Temperature Gradient Along the Plate
$T_w - T_\infty = 95\ C$

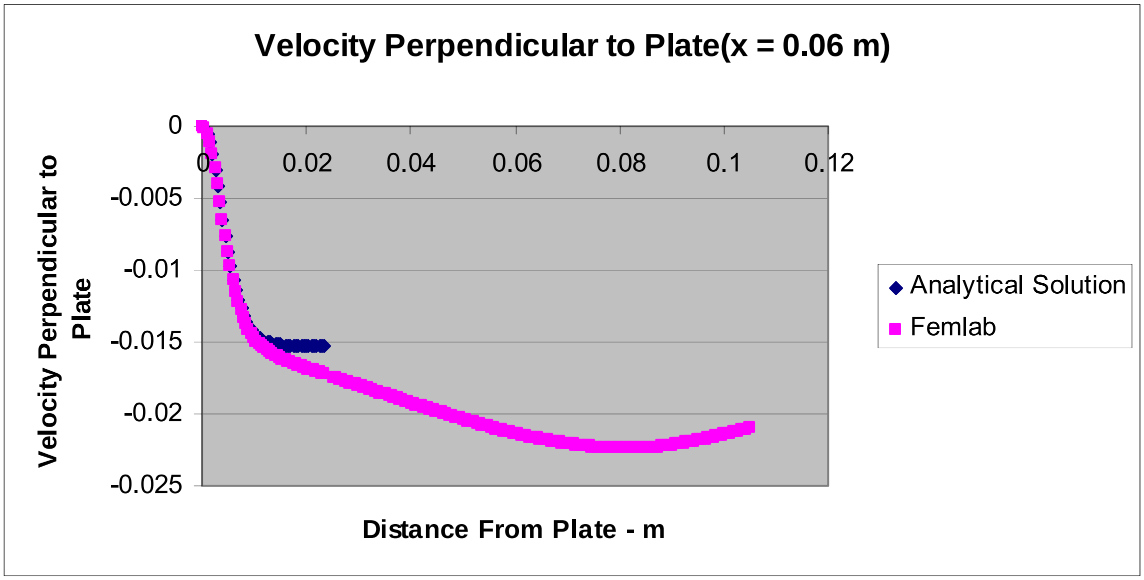


Figure 12. Velocity Perpendicular to Plate

Conclusions

There is very good agreement between Femlab, the analytical solution and data for the selected values of T_w, T_∞ and x (0.06 m). The boundary conditions selected were chosen so that the Femlab solution would have similar values to the analytical solution far from the plate The v velocity matches very well near the plate but varies from -0.0153 far from the plate (where the boundary layer equations may not be valid). The u velocity, however, does tend to zero far from the plate. In practice it is difficult to specify appropriate boundary conditions.(Bejan, 1995).

The solution (both Femlab and the Analytical solution) is dependent on the temperature chosen to evaluate the physical properties. Boelter, 1948 (p XII -30) remarks that analytical results check well with the experimental results of Schmidt and Beckmann, 1930 if the properties are chosen at T_w. This was difficult to verify since the data of McAdams could only be determined approximately since it was read from a chart. The data in McAdams are taken from Schmidt , 1928.

A model's documentation can be obtained by File➔Generate Report.

Nomenclature

English

c_p	specific heat, J/(kg K)
g	gravitational constant,m/s^2
Gr_x	Grashof Number = g $(T_w - T_\infty)$ $x^3/(\upsilon^2 * T_\infty)$
k	thermal conductivity, J/(s m K)
p_1	= ξ’
p2	= ξ’’
Pr	Prandtl Number = $c_p \mu /k$
q	= θ’
T	temperature, K
T_w	isothermal temperature of the plate, K
T_∞	temperature far from the plate, K
u	velocity in x – direction, m/s
v	velocity in y direction, m/s
x	distance measured from bottom of plate, upward, m
xx	distance measured from bottom of Femlab simulation, m
y	distance measured from lower edge of plate to right, m

Greek

α	thermal diffusivity, k / (ρ c_p) - m^2/s
β	thermal expansion coefficient, $= 1/T_\infty$ for ideal gas
η	independent variable
θ	dimensionless temperature = (T - T_∞ }/(T_w - T_∞)
μ	dynamic viscosity kg/(m s)
$\xi((\eta)$	function in similarity transformation
ρ	density, kg/m^3
υ	kinematic viscosity, m^2/s = μ / ρ
ψ	stream function

Acknowledgement

The help and comments of Comsol technical support and Dr. P. Rony are appreciated

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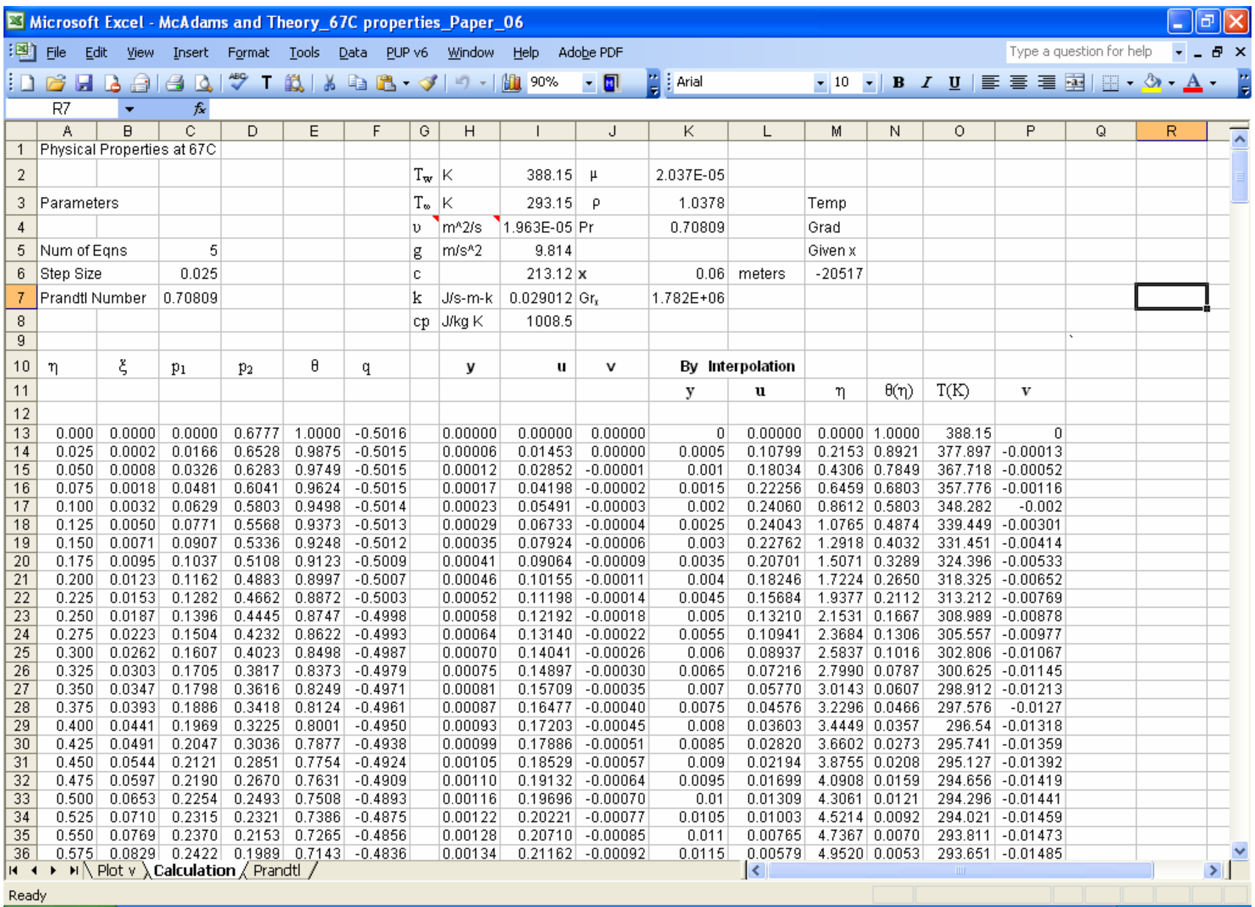
Appendix A

Spreadsheet Calculation



Microsoft Excel - McAdams and Theory_67C properties_Paper_06

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Physical Properties at 67C															
2							T_w	K	388.15	μ	2.037E-05					
3	Parameters						T_∞	K	293.15	ρ	1.0378		Temp			
4							υ	m^2/s	1.963E-05	Pr	0.70809		Grad			
5	Num of Eqns		5				g	m/s^2	9.814				Given x			
6	Step Size		0.025				c		213.12	x	0.06	meters	-20517			
7	Prandtl Number		0.70809				k	J/s-m-k	0.029012	Gr_x	1.782E+06					
8							cp	J/kg K	1008.5							
9																
10	η	ξ	p_1	p_2	θ	q		y	u	v	By Interpolation					
11											y	u	η	θ(η)	T(K)	v
12																
13	0.000	0.0000	0.0000	0.6777	1.0000	-0.5016		0.00000	0.00000	0.00000	0	0.00000	0.0000	1.0000	388.15	0
14	0.025	0.0002	0.0166	0.6528	0.9875	-0.5015		0.00006	0.01453	0.00000	0.0005	0.10799	0.2153	0.8921	377.897	-0.00013
15	0.050	0.0008	0.0326	0.6283	0.9749	-0.5015		0.00012	0.02852	-0.00001	0.001	0.18034	0.4306	0.7849	367.718	-0.00052
16	0.075	0.0018	0.0481	0.6041	0.9624	-0.5015		0.00017	0.04198	-0.00002	0.0015	0.22256	0.6459	0.6803	357.776	-0.00116
17	0.100	0.0032	0.0629	0.5803	0.9498	-0.5014		0.00023	0.05491	-0.00003	0.002	0.24060	0.8612	0.5803	348.282	-0.002
18	0.125	0.0050	0.0771	0.5568	0.9373	-0.5013		0.00029	0.06733	-0.00004	0.0025	0.24043	1.0765	0.4874	339.449	-0.00301
19	0.150	0.0071	0.0907	0.5336	0.9248	-0.5012		0.00035	0.07924	-0.00006	0.003	0.22762	1.2918	0.4032	331.451	-0.00414
20	0.175	0.0095	0.1037	0.5108	0.9123	-0.5009		0.00041	0.09064	-0.00009	0.0035	0.20701	1.5071	0.3289	324.396	-0.00533
21	0.200	0.0123	0.1162	0.4883	0.8997	-0.5007		0.00046	0.10155	-0.00011	0.004	0.18246	1.7224	0.2650	318.325	-0.00652
22	0.225	0.0153	0.1282	0.4662	0.8872	-0.5003		0.00052	0.11198	-0.00014	0.0045	0.15684	1.9377	0.2112	313.212	-0.00769
23	0.250	0.0187	0.1396	0.4445	0.8747	-0.4998		0.00058	0.12192	-0.00018	0.005	0.13210	2.1531	0.1667	308.989	-0.00878
24	0.275	0.0223	0.1504	0.4232	0.8622	-0.4993		0.00064	0.13140	-0.00022	0.0055	0.10941	2.3684	0.1306	305.557	-0.00977
25	0.300	0.0262	0.1607	0.4023	0.8498	-0.4987		0.00070	0.14041	-0.00026	0.006	0.08937	2.5837	0.1016	302.806	-0.01067
26	0.325	0.0303	0.1705	0.3817	0.8373	-0.4979		0.00075	0.14897	-0.00030	0.0065	0.07216	2.7990	0.0787	300.625	-0.01145
27	0.350	0.0347	0.1798	0.3616	0.8249	-0.4971		0.00081	0.15709	-0.00035	0.007	0.05770	3.0143	0.0607	298.912	-0.01213
28	0.375	0.0393	0.1886	0.3418	0.8124	-0.4961		0.00087	0.16477	-0.00040	0.0075	0.04576	3.2296	0.0466	297.576	-0.0127
29	0.400	0.0441	0.1969	0.3225	0.8001	-0.4950		0.00093	0.17203	-0.00045	0.008	0.03603	3.4449	0.0357	296.54	-0.01318
30	0.425	0.0491	0.2047	0.3036	0.7877	-0.4938		0.00099	0.17886	-0.00051	0.0085	0.02820	3.6602	0.0273	295.741	-0.01359
31	0.450	0.0544	0.2121	0.2851	0.7754	-0.4924		0.00105	0.18529	-0.00057	0.009	0.02194	3.8755	0.0208	295.127	-0.01392
32	0.475	0.0597	0.2190	0.2670	0.7631	-0.4909		0.00110	0.19132	-0.00064	0.0095	0.01699	4.0908	0.0159	294.656	-0.01419
33	0.500	0.0653	0.2254	0.2493	0.7508	-0.4893		0.00116	0.19696	-0.00070	0.01	0.01309	4.3061	0.0121	294.296	-0.01441
34	0.525	0.0710	0.2315	0.2321	0.7386	-0.4875		0.00122	0.20221	-0.00077	0.0105	0.01003	4.5214	0.0092	294.021	-0.01459
35	0.550	0.0769	0.2370	0.2153	0.7265	-0.4856		0.00128	0.20710	-0.00085	0.011	0.00765	4.7367	0.0070	293.811	-0.01473
36	0.575	0.0829	0.2422	0.1989	0.7143	-0.4836		0.00134	0.21162	-0.00092	0.0115	0.00579	4.9520	0.0053	293.651	-0.01485

Plot v | Calculation | Prandtl

Ready

VBA Macro

```
Option Explicit

Private Function Integ(x, y, prm)
Application.Volatile True

Dim N, IR, nn, I As Integer
Dim h, xx As Single

N = prm(1)

nn = N + 1

ReDim yy(1 To N) As Single
ReDim ddd(1 To nn)

h = prm(2)

xx = x

For I = 1 To N
   yy(I) = y(I)
Next

IR = rk4a(N, h, xx, yy, prm)

xx = xx + h

ddd(1) = xx

For I = 2 To nn
   ddd(I) = yy(I - 1)
Next I

Integ = ddd


End Function
```

```
Public Function rk4a(N, h, x, y, prm)
'
'Modified from Pedro L. Claveria abril/2002
'based in EMR Technology Group Library
'
'
'n = number of equations
'h  = step size for integration
'x = independent variable
'y = vector of dependent variables
'prm = vector parameters

'MsgBox "Entering rk4a" & x & "  " & h

ReDim ccc(N), fff(N)
ReDim k1(N), k2(N), k3(N), k4(N)
ReDim y2(N), y3(N), y4(N)

Dim muda1, muda2, muda3, muda4 As Single
Dim I As Integer

'Calculation of k1
muda1 = dydx(x, y, prm, fff)
For I = 1 To N: k1(I) = fff(I): Next
'Calculation of k2
For I = 1 To N: y2(I) = y(I) + 0.5 * h * k1(I): Next
muda2 = dydx(x + h / 2, y2, prm, fff)
For I = 1 To N: k2(I) = fff(I): Next
'Calculation of k3
For I = 1 To N: y3(I) = y(I) + 0.5 * h * k2(I): Next
muda3 = dydx(x + h / 2, y3, prm, fff)
For I = 1 To N: k3(I) = fff(I): Next
'Calculation of k4
For I = 1 To N: y4(I) = y(I) + h * k3(I): Next
muda4 = dydx(x + h, y4, prm, fff)
For I = 1 To N: k4(I) = fff(I): Next

'New values of the dependent variables
For I = 1 To N
   ccc(I) = y(I) + (h / 6) * (k1(I) + 2 * k2(I) + 2 * k3(I) + k4(I))
Next I


For I = 1 To N
   y(I) = ccc(I)
Next I

rk4a = 0

End Function
```

```
Private Function dydx(x, y, prm, fff)
Dim Pr As Single

Pr = prm(3)
fff(1) = y(2)
fff(2) = y(3)
fff(3) = -3 * y(1) * y(3) + 2 * (y(2) ^ 2) - y(4)
fff(4) = y(5)
fff(5) = -3 * Pr * y(1) * y(5)

dydx = 0
End Function
```