# A Review: The Pendulum and VBA™

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## Introduction

The motion of a simple pendulum in Cartesian coordinates (x, y) with center (0,0) and length L has received considerable attention [1,2,3]. Although a number of codes [4] to solve differential algebraic equation systems (DAEs) have been developed, none have been found to be implemented in VBA™.

This paper reviews the development of the pendulum equations and presents VBA™ coding for the solution of the equations. A fourth order Runge-Kutta integration routine is utilized to integrate two dependent variables.

Excel routine in ***Microsoft Office Home and Student 2010*** was used in this study.

## Equation Development

Figure 1 depicts a simple pendulum. In addition to the equations [2] for Newton's Law (force = mass x acceleration) there is an equality constraint that must be satisfied:

$$x(t)^2 + y(t)^2 = L^2 \quad \text{(t is time, L is the length of the rod)} \qquad (1)$$

$$m \cdot d^2/dt^2\, x(t) = (x(t)/L) \cdot F(t) \quad \text{(x component of the tension force)}$$

$$m \cdot d^2/dt^2\, y(t) = -m \cdot g + (y(t)/L) \cdot F(t) \quad \text{(the y-component of the tension force and the additional downward force due to gravity (g) acting on the pendulum)}$$

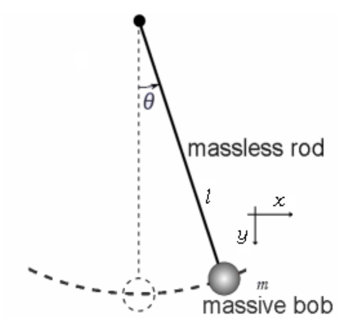


**Figure 1. The simple pendulum[5]**

Dividing by m, substituting λ (t) = F(t)/ (L· m) and dropping the t in λ (t), x(t), and y(t) yields

$$x^2 + y^2 = L^2 \qquad (2)$$
$$x'' = \lambda x$$
$$y'' = -g + \lambda y$$

Finally letting u = x' and v = y' there results

$$x^2 + y^2 = L^2 \qquad (3)$$

$$x' = u$$
$$y' = v$$
$$u' = \lambda x$$
$$v' = -g + \lambda y$$

The number of times it takes to reduce the dependent variables to a set of first order ordinary differential equations determines the index of the system [5]. Here λ is a dependent variable whose derivative is sought.

Differentiate the equality constraint once (First time):

$$xx' + yy' = 0$$

Substitute for x',y'

$$xu + yv = 0$$

Differentiate the constraint again (Second time):

$$xu' + yv' + u^2 + v^2 = 0$$

Substitute for u', v'

$$x^2\lambda + y^2\lambda - yg + u^2 + v^2 = 0$$

Substitute the original constraint:

$$L^2\lambda - yg + u^2 + v^2 = 0$$

Differentiate the constraint again and solve for λ' (Third time):

$$\lambda' = (1/L^2)\,(vg - 2\lambda ux - 2v(\lambda y - g))$$

Using $xu + yv = 0$

$$\lambda' = (1/L^2)\,.\,(3vg)$$

This then is an index 3 system:

In this paper λ is a dimensionless measure of the tension force. When the pendulum equations are developed using the Euler-Lagrange equation, λ is defined as a Lagrange multiplier [6,7,8].

**Solving the Differential Equations**

Combining the above equations the system to be solved is a semi-explicit DAE of index 1[4] :

$$x' = u$$
$$u' = \lambda x$$
$$0 = x^2 + y^2 - L^2$$
$$0 = ux + vy$$
$$0 = u^2 - gy + v^2 + L^2\lambda$$

Consistent starting values can be set by considering the pendulum in a vertical position (x=0, ϴ=0). If the value of u is estimated, then both x and u can be integrated if the sign of y is given in $y = \pm\sqrt{(L^2 - x^2)}$.

Again if $y \neq 0$ then $v = -ux/y$ and $\lambda = (gy-u^2 - v^2)/L^2$. To proceed to the next point it is sufficient to get the derivatives of x and u. ($x' = u$, $u' = \lambda x$).

The integration is carried out stepwise with a fourth order Runge-Kutta code in VBA procedure INTEG (Appendix 1). Function dy/dx evaluates the right hand side of the differential equations for x and u.

Table 1 is the spreadsheet of the solution. The initial values of x and u are set: of x = 0 and u = 4.4. The 4.4 was chosen (by trial) so that the angle of deflection (Θ) would get as close as possible to horizontal from the starting position. (The value of $\Theta_{max}$ is linearly dependent on u). The values of y, v, and λ are subsequently calculated. The value of Θ (in degrees) is determined from the value of ATN(x/y) in radians.

The calling sequence to Function INTEG is:

INTEG (current time, current values of x and u, parameter vector)

Where:

**Parameter Vector**

| | | | |
|---|---|---|---|
| 1 | N | 2 | Number of dependent variables |
| 2 | h | 0.01 | step size in sec |
| 3 | g | 9.806650 | standard acceleration of gravity $m/sec^2$ |
| 4 | L | 1 | length of rod, m |
| 5 | m | 1 | mass of pendulum, kg |

**Figure 2. The parameter vector**

Typically if the initial values of time, x and u are in row 7 of the spreadsheet, then the values at the incremented time are specified in row 8 by highlighting A8 to C8 and entering:

= INTEG($A7, $B7:$C7,$I$6) with **Ctrl+Shift+Enter** (Array formula):

The A column is the time, columns B and C are the dependent variables x and u and $I$6 is the starting point of the parameter vector column.

The values at the incremented time, x and u then appear in row 8. Values of y, v, λ, x/y and Θ are calculated from the new values of x and u. Copies of row 8 are then made. The integration is carried out for 3.93 seconds.

## Pendulum Study

| A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|
| Time | x | u | y | v | λ | x/y | Θ | prm |
| sec | | | | | | | | 2.000000 |
| 0.00 | 0.0000 | 4.4000 | -1.0000 | 0.0000 | -29.1667 | 0.0000 | 0.0000 | 0.010000 |
| 0.01 | 0.0440 | 4.3936 | -0.9990 | 0.1934 | -29.1382 | -0.0440 | -2.5206 | 9.806650 |
| 0.02 | 0.0878 | 4.3744 | -0.9961 | 0.3857 | -29.0530 | -0.0882 | -5.0387 | 1.000000 |
| 0.03 | 0.1314 | 4.3426 | -0.9913 | 0.5757 | -28.9115 | -0.1326 | -7.5519 | 1.000000 |
| 0.04 | 0.1746 | 4.2985 | -0.9846 | 0.7624 | -28.7145 | -0.1774 | -10.0577 | |
| 0.05 | 0.2174 | 4.2425 | -0.9761 | 0.9447 | -28.4633 | -0.2227 | -12.5538 | |
| 0.06 | 0.2595 | 4.1750 | -0.9658 | 1.1216 | -28.1592 | -0.2687 | -15.0375 | |
| 0.07 | 0.3008 | 4.0966 | -0.9537 | 1.2922 | -27.8040 | -0.3154 | -17.5068 | |
| 0.08 | 0.3413 | 4.0079 | -0.9399 | 1.4555 | -27.3996 | -0.3632 | -19.9591 | |
| 0.09 | 0.3809 | 3.9098 | -0.9246 | 1.6109 | -26.9483 | -0.4120 | -22.3922 | |
| 0.10 | 0.4195 | 3.8029 | -0.9077 | 1.7575 | -26.4526 | -0.4621 | -24.8040 | |
| 0.11 | 0.4570 | 3.6881 | -0.8895 | 1.8948 | -25.9151 | -0.5138 | -27.1922 | |
| 0.12 | 0.4933 | 3.5663 | -0.8699 | 2.0222 | -25.3387 | -0.5670 | -29.5547 | |
| 0.13 | 0.5283 | 3.4384 | -0.8491 | 2.1394 | -24.7263 | -0.6222 | -31.8895 | |
| 0.14 | 0.5620 | 3.3054 | -0.8271 | 2.2459 | -24.0809 | -0.6795 | -34.1946 | |
| 0.15 | 0.5944 | 3.1681 | -0.8042 | 2.3415 | -23.4059 | -0.7391 | -36.4681 | |
| 0.16 | 0.6254 | 3.0275 | -0.7803 | 2.4262 | -22.7043 | -0.8014 | -38.7083 | |
| 0.17 | 0.6549 | 2.8844 | -0.7557 | 2.4997 | -21.9794 | -0.8666 | -40.9133 | |
| 0.18 | 0.6830 | 2.7398 | -0.7304 | 2.5623 | -21.2345 | -0.9352 | -43.0816 | |
| 0.19 | 0.7097 | 2.5946 | -0.7045 | 2.6138 | -20.4728 | -1.0074 | -45.2114 | |
| 0.20 | 0.7349 | 2.4495 | -0.6781 | 2.6546 | -19.6976 | -1.0837 | -47.3014 | |

**Table 1. Spreadsheet for Solving the Differential Equations (abbreviation of table to 3.93 sec)**

## Graphing the Results

Figures 3-6 are plots generated from the spreadsheet. Figure 3 shows that the value of y approaches zero as the pendulum becomes horizontal. Figure 4 records the swing of the pendulum. Figure 5 implies the angle of the pendulum between the vertical and the horizontal. Note that ATAN(x/y) gives the angle in radians. Figure 6 documents λ, the dimensionless tension in the rod

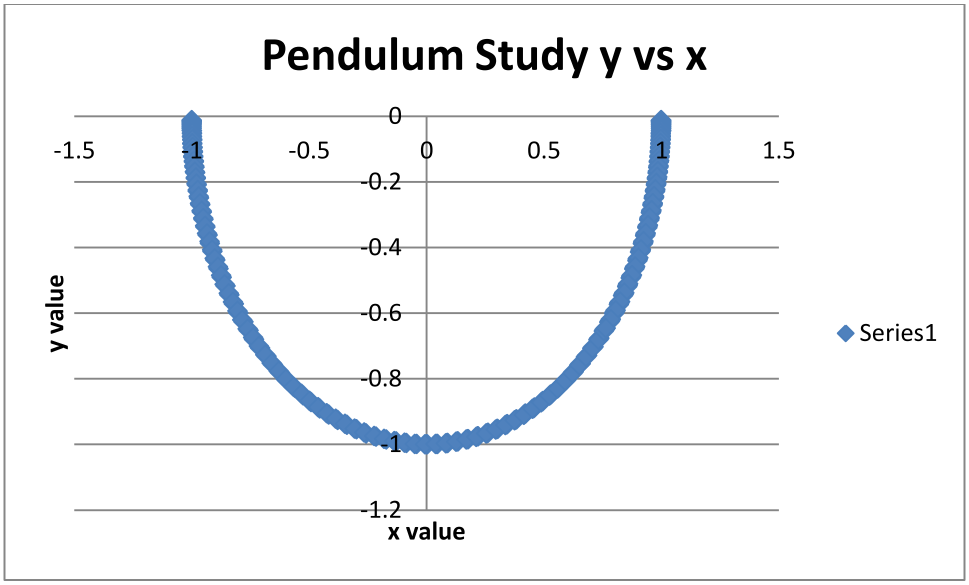


**Figure 3. y vs x**

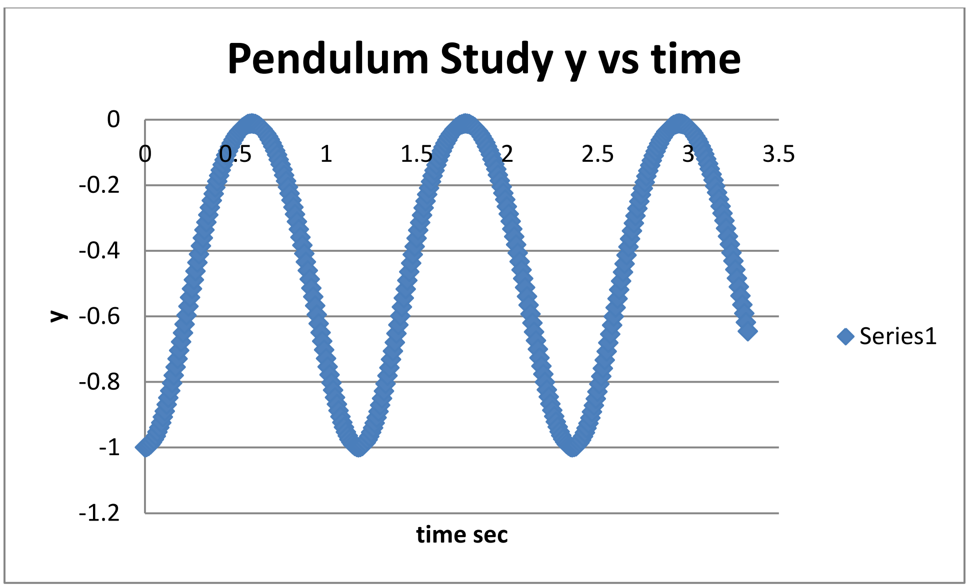


**Figure 4. y vs time**

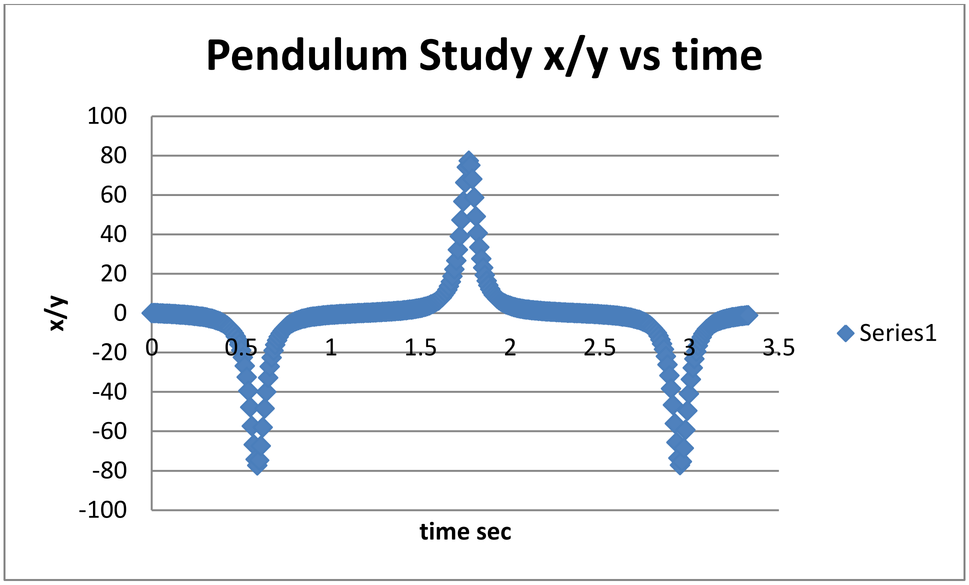


**Figure 5. x/y vs time**

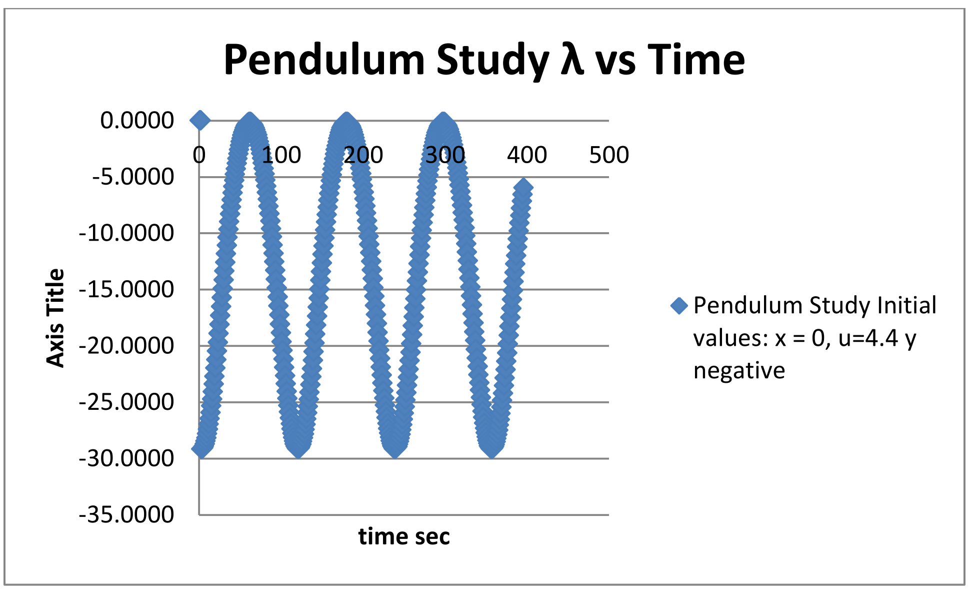


**Figure 6. λ vs time**

### Determining the Period

The period of the pendulum can be determined from the spreadsheet. In Table 2 are extractions from the spreadsheet at the point the pendulum has swung to its maximum angle. The elapsed time is 2.95 – 0.59 = 2.36 sec.

| Time sec | x | u | y | v | λ | x/y | Θ |
|---|---|---|---|---|---|---|---|
| 0.58 | 0.9999 | 0.00136 | -0.0134 | 0.1013 | -0.1421 | -74.383 | -89.2298 |
| **0.59** | **0.9999** | **4.2E-05** | **-0.0129** | **0.0032** | **-0.1267** | **-77.392** | **-89.2597** |
| 0.6 | 0.9999 | -0.00127 | -0.0134 | -0.095 | -0.1402 | -74.743 | -89.2335 |
| | | | | | | | |
| 2.94 | 0.9999 | 0.00155 | -0.0136 | 0.1142 | -0.1463 | -73.59 | -89.2215 |
| **2.95** | **0.9999** | **0.00021** | **-0.0129** | **0.0162** | **-0.1271** | **-77.298** | **-89.2588** |
| 2.96 | 0.9999 | -0.00109 | -0.0133 | -0.082 | -0.1368 | -75.383 | -89.24 |

**Table 2. Extractions from the spreadsheet at times of maximum angles ($\Theta_{max}$) from the vertical**

The period from the spreadsheet can be compared with that from the literature [7].
Table 3 evaluates the period (Equation (4)) for a pendulum whose maximum angle ($\Theta_{rad}$) is 1.558.

$$T = 2 * Pi * \sqrt{\frac{L}{g}} \left( 1 + \frac{1}{16} * \Theta_{rad}^{2} + \frac{11}{3072} * \Theta_{rad}^{4} + \frac{173}{737280} * \Theta_{rad}^{6} + \frac{22931}{1321205760} * \Theta_{rad}^{8} + \dots + \frac{1319183}{951268147200} * \Theta_{rad}^{10} + \dots \right) \quad (4)$$

**Period Calculation**

| | | |
|---|---|---|
| Pi | 3.141592654 | |
| L | 1 | |
| g | 9.80665 | |
| 2*Pi*SQRT(L/g) | 2.006409293 | |
| | | |
| $\Theta_{rad}$ | 1.558 | 89.26 Degrees |
| Term in Series | | |
| 1 | 1 | |
| 2 | 0.151687054 | |
| 3 | 0.021091549 | |
| 4 | 0.003354433 | |
| 5 | 0.00060218 | |
| 6 | 0.000116774 | |
| | | |
| Sum | 1.17685199 | |
| | | |
| T | 2.361246769 | |

**Table 3. Evaluation of period for maximum angle $\Theta_{max}$**

The period from the spreadsheet matches that from the literature series very well (2.36 sec).

## Conclusions

Index reduction (by differentiation of the constraint) reduces the index 3 pendulum equations to a semi-explicit DAE of index 1. The equations are then integrated with an ODE code (Runge-Kutta) written in VBA™.

The accuracy of the integration is confirmed by comparing a spreadsheet period with a literature value.

Numerical instability is not observed.

The equations are limited to angles up to 90$^o$ from the vertical at which point y goes to zero.

## References


1. Petzold, L., *Numerical Solution of Differential-Algebraic-Equations*
   University of Minnesota, Minneapolis, Minnesota, 55455
2. Quicksheets - *Calculus and Differential Equations*
   DAE- *Simple Pendulum Motion*
3. Tan, Suri, *Differential-Algebraic Equations (DAEs) and numerical Methods,*
   http://www.cfm.brown.edu/people/jansh/page5/page10/page40/assets/Sirui_Talk.pdf
4. *Differential Algebraic Equation* - Wikipedia the free encyclopedia
5. *The Differential-Algebraic Equation (DAE) Solver*
   http://ww.ni.com/example/31306/en
6. MIT Open Couse Ware, http://ocw.mit.edu
7. Pendulum (mathematics) – Wikipedia the free encyclopedia
8. Schulz, S., *Four Lectures on Differential-Algebraic-Equations,*
   Humboldt Universitat zu Berlin, June 13, 2003
   https://www.math.auckland.ac.nz/deptdb/dept-reports/497.pdf


## Nomenclature

F tension force in rod - newtons
g gravitational acceleration – m/sec$^2$
l, L Length of pendulum - meters
m mass of the pendulum’s mass point – kg
T period of pendulum - sec
u equal to x’ (momentum variable)
v equal to y’ (momentum variable)
x Cartesian coordinate horizontal – meters
x’ first derivative of x
x’’ second derivative of x
y Cartesian coordinate vertical- meters
y’ first derivative of y
y’’ second derivative of y

Θ angle of deflection measured from downward position of the rod to the x horizontal positive Axis – Degrees

$\Theta_{max}$ Amplitude or largest angle achieved – Degrees

$\Theta_{rad}$ Amplitude or largest angle achieved - Radians

λ dimensionless constraint force

**Other**

DAE Differential-Algebraic Equation

ODE Ordinary Differential Equation: ODE's have index of zero

Semi-explicit DAE of index 1: An ODE with constraints

VBA™ Visual Basic for Applications

## Appendix 1. Listing of Procedure INTEG

```
Option Explicit

Private Function integ(x, y, prm)

Dim N, IR, NN, I As Integer
Dim h, xx As Double

N = prm(1)

NN = N + 1

ReDim yy(1 To N) As Double
ReDim ddd(1 To NN)

h = prm(2)
xx = x

For I = 1 To N
  yy(I) = y(I)
Next

IR = rk4a(N, h, xx, yy, prm)

xx = xx + h

ddd(1) = xx
```

```
For I = 2 To NN
   ddd(I) = yy(I - 1)
Next I

integ = ddd


End Function




Public Function rk4a(N, h, x, y, prm)
'
'Modified from Pedro L. Claveria abril/2002
'based in EMR Technology Group Library
'
'
'n = number of equations
'h  = step size for integration
'x = independent variable
'y = vector of dependent variables
'prm = vector parameters



ReDim ccc(N), fff(N)
ReDim k1(N), k2(N), k3(N), k4(N)
ReDim y2(N), y3(N), y4(N)

Dim muda1, muda2, muda3, muda4 As Double
Dim I As Integer

'Calculation of k1
muda1 = dydx(x, y, prm, fff)
For I = 1 To N: k1(I) = fff(I): Next
'Calculation of k2
For I = 1 To N: y2(I) = y(I) + 0.5 * h * k1(I): Next
muda2 = dydx(x + h / 2, y2, prm, fff)
For I = 1 To N: k2(I) = fff(I): Next
'Calculation of k3
For I = 1 To N: y3(I) = y(I) + 0.5 * h * k2(I): Next
muda3 = dydx(x + h / 2, y3, prm, fff)
For I = 1 To N: k3(I) = fff(I): Next
'Calculation of k4
```

```
For I = 1 To N: y4(I) = y(I) + h * k3(I): Next
muda4 = dydx(x + h, y4, prm, fff)
For I = 1 To N: k4(I) = fff(I): Next

'New values of the dependent variables
For I = 1 To N
   ccc(I) = y(I) + (h / 6) * (k1(I) + 2 * k2(I) + 2 * k3(I) + k4(I))
Next I

For I = 1 To N
   y(I) = ccc(I)
Next I

rk4a = 0

End Function

Private Function dydx(xx, yy, prm, fff)

Dim xxx, u, v, g, y, lam, L As Double

xxx = yy(1)
u = yy(2)

g = prm(3)
L = prm(4)

' Work on fff(1)
' Calculation of lam

y = -Sqr(L ^ 2 - xxx ^ 2)
v = -u * xxx / y

lam = (g * y - u ^ 2 - v ^ 2) / L ^ 2

fff(1) = u
fff(2) = lam * xxx

dydx = 0
End Function
```