On Similarity Transformations
A Classical Approach

Edward M. Rosen
EMR Technology Group

Introduction

Similarity transformations are often utilized to convert partial differential equations to a set of ordinary differential equations [1]. The ordinary differential equations may then be solved numerically [2,3] or a closed form solution may possibly be found.

Subramanian [4] mentions that students often have difficulties in carrying out the transformation. The reason for this may be that the derivation of the transform is long and tedious. It may also be that the procedures for the derivation are not well known or may not be easily retrieved from the literature. He suggests the use of Maple [5] to aid the student.

It the purpose of this paper to detail the use of one such transformation and show how the development of another is carried out without the use of a computer algebra system.

A Simple Transformation

Consider the transit heat-conduction problem in a slab. The applicable equations and boundary conditions are:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$\begin{aligned} u(x,0) &= 0 \\ u(0,t) &= 1 \text{ and } u(\infty,t) = 0 \end{aligned} \tag{2}$$

Assume following transformation variable is known:

$$\eta = \frac{x}{2\sqrt{\alpha t}} \tag{3}$$

To implement the transformation determine the following:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \eta}\frac{x}{2\sqrt{\alpha}}(-1/2)\,t^{-3/2}$$

$$= \ - \frac{1}{2t} \frac{x}{2\sqrt{\alpha t}} \frac{\partial u}{\partial \eta}$$

$$= -\eta \frac{1}{2t} \frac{\partial u}{\partial \eta} \qquad (4)$$

Also**:**

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{1}{2\sqrt{\alpha t}} \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{1}{2\sqrt{\alpha t}} \ \frac{\partial u}{\partial \eta} \right)$$

$$= \frac{1}{2\sqrt{\alpha t}} \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial x}$$

$$= \frac{1}{2\sqrt{\alpha t}} \frac{\partial^2 u}{\partial \eta^2} \frac{1}{2\sqrt{\alpha t}} \qquad (5)$$

Substituting Equations (4) and (5) into Equation (1)

$$- \frac{1}{2t} \eta \frac{\partial u}{\partial \eta} = \alpha \left(\frac{1}{2\sqrt{\alpha t}} \frac{\partial^2 u}{\partial \eta^2} \frac{1}{2\sqrt{\alpha t}} \right)$$

Simplifying:

$$\frac{1}{2} \frac{\partial^2 u}{\partial \eta^2} + \eta \frac{\partial u}{\partial \eta} = 0$$

Since u is only a function of η (t and x has been eliminated) u(x, t) can be replaced by U(η):

$$\frac{1}{2} \frac{d^2 U}{d\eta^2} + \eta \frac{dU}{d\eta} = 0 \qquad (6)$$

By substituting the boundary conditions of Equations (2) into η the new boundary conditions become:

$$U(0) = 1$$

$$U(\infty) = 0 \qquad (7)$$

A solution to Equation (6) subject to Equations (7) is known to be [4]

$$U(\eta) = \operatorname{erfc}(\eta)$$

In terms of the original variables:

$$u(x, t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

Finding a Transformation

The equations for the free convection from a vertical flat plate have been extensively studied [2,3,6]**.** The applicable equations are [6]

Continuity
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (8)$$

Hydrodynamic
$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \upsilon\,\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) \qquad (9)$$

Energy
$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial y^2} \qquad (10)$$

In terms of the dimensionless variable $\theta = \dfrac{T - T_\infty}{T_w - T_\infty}$

Continuity
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (11)$$

Hydrodynamic
$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \upsilon\,\frac{\partial^2 u}{\partial y^2} + g\,\beta\,(T_w - T_\infty)\,\theta \qquad (12)$$

Energy $$u\,\frac{\partial\theta}{\partial x}+v\,\frac{\partial\theta}{\partial y}=\alpha\,\frac{\partial^2\theta}{\partial y^2} \tag{13}$$

Boundary conditions:

y = 0: u = 0, v = 0. T = T_w

y = ∞: u = 0, v = 0, T = T_∞

Physical properties are assumed constant

In order to develop a transformation proceed as follows[6]:

Introduce a stream function ψ by putting u = $\frac{\partial\psi}{\partial y}$ and v = -$\frac{\partial\psi}{\partial x}$ and substitute into Equations (11 -13):

$$\frac{\partial^2\psi}{\partial x\,\partial y}-\frac{\partial^2\psi}{\partial x\partial y}=0 \tag{14}$$

$$\frac{\partial\psi}{\partial y}\,\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\,\frac{\partial^2\psi}{\partial y^2}=\upsilon\,\frac{\partial^3\psi}{\partial y^3}+g\;\beta(T_w-T_\infty)\theta \tag{15}$$

$$\frac{\partial\psi}{\partial y}\,\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\,\frac{\partial\theta}{\partial y}=\alpha\,\frac{\partial^2\theta}{\partial y^2} \tag{16}$$

Now assume that:

$$\psi(x,y)=b\;x^n\;\xi(\eta) \tag{17}$$

and $$\eta=\frac{cy}{x^p} \tag{18}$$

Then

$$\frac{\partial\eta}{\partial y}=\frac{c}{x^p} \tag{19}$$

and $$\frac{\partial\eta}{\partial x}=-cyp\;x^{-(p+1)} \tag{20}$$

Calculate the following derivatives noting that ξ is a function of η:

A. $\dfrac{\partial \psi}{\partial y} = bx^n \ \xi' \ \dfrac{c}{x^p}$

B. $\dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi}{\partial y}\right) = \dfrac{\partial\left(\dfrac{\partial \psi}{\partial y}\right)}{\partial \eta} \dfrac{\partial \eta}{\partial y} = bx^n \left(\dfrac{c}{x^p}\right)^2 \ \xi''$

C. $\dfrac{\partial^3 \psi}{\partial y^3} = bx^n \left(\dfrac{c}{x^p}\right)^3 \ \xi'''$

D. $\dfrac{\partial \psi}{\partial x} = bx^n \ \xi' \ \left(-c \ y \ p \ x^{-\{p+1)}\right) + \xi \ b \ n \ x^{n-1}$

E. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial \psi}{\partial y}\right) = \xi' \ (bc)(n-p)\, x^{n-p-1} + (bc)\, x^{n-p} \ \xi'' \ \left(-c \ y \ p \ x^{-(p+1}\right)$

Substitute A, B, C, D, E into Equation (15) , simplify and divide by b^2c^2:

$$(n-p)\ x^{2n-2p-1} \ \xi'^2 - n \ x^{2n-2p-1} \ \xi \ \xi'' = \nu \ x^{n-3p} \left(\frac{c}{b}\right) \xi''' + \frac{g \ \beta(T_w - T_\infty)}{b^2 \ c^2} \ \theta \qquad (21)$$

If x is to disappear from Equation (21) the following must hold:

2n – 2p – 1 = 0

n – 3p = 0

Solve for n and p:

$$n = \frac{3}{4} \qquad p = \frac{1}{4}$$

Substitute the values of n and p into Equation (21) and multiply through by 4:

$$2\left(\xi'\right)^2 - 3\,\xi\,\xi'' = \frac{\nu 4c}{b}\,\xi''' + \frac{4g\beta\left(T_w - T_\infty\right)}{b^2c^2} \tag{22}$$

Since c and b are arbitrary, let

$$\frac{4c}{b}\,\nu = 1 \qquad and \qquad \frac{4g\beta\left(T_w - T_\infty\right)}{c^2\,b^2} = 1$$

Solve for b and c:

$$b = 4\,\upsilon\,c \qquad \text{and} \qquad c = \sqrt[4]{\frac{g\beta\left(T_w - T_\infty\right)}{4\nu^2}}$$

Equation (22) becomes

$$\xi''' + \theta + 3\,\xi\,\xi'' - 2\left(\xi'\right)^2 = 0 \tag{23}$$

Proceed in a similar manner for θ which is assumed to be only a function of η:

F: $$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial\eta}\left(\frac{\partial\eta}{\partial x}\right) = \theta'\;c\;y\,(-p)\;x^{-(p+1)}$$

G: $$\frac{\partial\theta}{\partial y} = \frac{\partial\theta}{\partial\eta}\left(\frac{\partial\eta}{\partial y}\right) = \theta'\left(\frac{c}{x^p}\right)$$

H: $$\frac{\partial^2\theta}{\partial y^2} = \theta^{"} \left(\frac{c}{x^p}\right)^2$$

Substitute Equations A, D and F, G, H and into Equation (16) and simplify:

$$-\xi\; b\; n\; x^{n-1}\;\theta^{'} = \alpha\;\theta^{"}\left(\frac{c}{x^p}\right)$$

Substitute for b, n and p and simplify:

$$3\;\mathrm{Pr}\;\xi\;\theta^{'} + \theta^{"} = 0 \qquad (24)$$

Boundary conditions are:

$$\xi = \xi^{'} = 0\; and\; \theta = 1\; at\; \eta = 0\; and\; \xi^{'} = 0, \theta = 0\; at\; \eta = \infty$$

The dependent variables u and v are then:

$$\mathrm{u} = \frac{\partial\psi}{\partial y} = 4\nu\; c^2\; x^{1/2}\;\xi^{'}$$

$$\mathrm{v} = -\frac{\partial\psi}{\partial x} = \nu\; c\; x^{-1/4}\;(\eta\;\xi^{'} - 3\;\xi)$$

The numerical solution to Equations (23) and (24) is discussed in [2].

Conclusions

The use of similarity transformations to convert partial differential equations to ordinary differential equations can be long (and tedious). However, the procedure is straightforward and can be carried out in a classical fashion.

References

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Nomenclature
(Free Convection Transformation)

English

c_p	specific heat, J/(kg K)
g	gravitational constant,m/s^2
k	thermal conductivity, J/(s m K)
Pr	Prandtl Number = $c_p \mu /k$
T	temperature, K
T_w	isothermal temperature of the plate, K
T_∞	temperature far from the plate, K

u	velocity in x – direction, m/s
v	velocity in y direction, m/s
x	distance measured from bottom of plate, upward, m
y	distance measured from lower edge of plate to right, m

Greek

α	thermal diffusivity, k / (ρ c_p) - m^2/s
β	thermal expansion coefficient, $= 1/T_\infty$ for ideal gas
η	independent variable
θ	dimensionless temperature = (T - T_∞ }/(T_w - T_∞)
μ	dynamic viscosity kg/(m s)
$\xi((\eta)$	function in similarity transformation
ρ	density, kg/m^3
υ	kinematic viscosity, $m^2/s = \mu / \rho$
ψ	stream function